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13. ABSTRACT (Maximum 200 words)

This final technical report contains the Grantee Data and technical results obtained during the grant period, May 1, 1996 to Dec. 31, 1999. General theoretical formulations of laminated composite plate structures with piezoelectric and magnetostrictive layers are developed and finite element analyses are developed. Constitutive models of ferro-electrics and shape memory alloys have been developed, and the results are not reported here as it is documented in a Ph.D. dissertation of G. Rengarajan. Vibration suppression of laminated plates with magnetostrictive layers is also studied and results are presented. The formulations developed and constitutive models studied should be of interest to industry as well government labs in gaining a better understanding of the technical issues in the analysis and design of smart structural systems.

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Computational Approaches for Smart Materials and Structural Systems

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Preface

This final technical report contains a summary of research carried out on Grant DAAH04-96-1-0080 from the Army Research Office (ARO) to Texas A&M University, with Dr. J. N. Reddy as the principal investigator. Following Grantee data on the project on pages ii–v, the remainder of the report is devoted to the technical discussion. To limit the size of the report to a reasonable number of pages, certain topics are not covered in detail. Additional information on constitutive modeling of shape memory alloys, electrostrictive and magnetostrictive materials and associated finite element formulations can be found in the dissertation of Dr. Govind Rengarajan (*On the Inelastic Behavior of Crystalline Solids*, Ph.D. Thesis, by Govind Rengarajan, Department of Mechanical Engineering, Texas A&M University, September 1998).

Table of Contents

Army Research Office (ARO) Grantee Data:

ARO Grant Number: DAAH04-96-1-0080

Grant Title: Computational Approaches for Smart Materials and Structural Systems

Grant Amount: $210,000

Grant Period: 05/01/1996 – 12/30/1999

Research Objectives:

- Develop computational strategies to study material behavior using coupled electrothermomechanical analysis.
- Develop constitutive models for martensitic phase transformations in shape memory alloys.
- Develop models of ferroelectric and electrostrictive behavior of single crystals and ceramics. Include nonlinearity between strain and electric field (electrostrictive), and nonlinearity and dissipation between polarization and electric field (ferroelectrics).
- Study of structures with embedded or mounted sensors and actuators.

Major Accomplishments:

- Development of multiple model computational approaches for efficient stress analyses.
- General formulations based on refined plate theories to study smart structural systems with sensors and actuators.
- Development of a general constitutive model for ferroelectrics (including the piezoelectrics) and electrostrictives.
- Development of a constitutive model for shape memory behavior.
- Computational analysis of shape memory behavior.
- Computational analysis of ferroelectric and electrostrictive behavior.
- Vibration suppression using magnetostrictive materials.

Impact of the Research Conducted:

The research carried out on this research grant is expected to aid in a better understanding of the technological issues in the design and development of smart materials and structural systems related to Army's needs, such as advanced rotorcrafts, weapon platforms, weapon/ordinance systems, etc. The research carried out in this area by the principal investigator and his colleagues have already finding interest and application among researchers and analysts at universities (University of Maryland, Georgia Tech, Virginia Tech), in industry (Lockheed - Georgia, Bell Helicopter Textron - Ft. Worth), and government laboratories (U.S. TACOM).

Some of the possible applications of the developed methodology are:

- Constitutive models for martensitic phase transformations in shape memory alloys. Coupled analysis of smart structures under varying conditions (numerical simulations).
- Study of structures with embedded or mounted sensors and actuators at different locations, with minimal restrictions on the geometry of the structure.
- Use of the methodologies developed herein to build software to aid the design process, eventually leading to a complete numerical simulation of the operating conditions of a smart structural system.
- Applications of the developed tool to design smart material systems related to army's needs in weapon systems, rotorcrafts, land vehicles, and so on.

Journal Papers Resulting from (at least partial support of) ARO Grant:

- D. H. Robbins and J. N. Reddy, "Variable Kinematic Modeling of Laminated Composite Plates", *Int. J. Numer. Meth. Engng.,* Vol. 39, pp. 2283-2317, 1996.
- D. H. Robbins and J. N. Reddy, "An Efficient Computational Model for the Stress Analysis of Smart Plate Structures", *Smart Materials & Structures,* Vol. 5, pp. 353–360, 1996.
- F. T. Kokkinos and J. N. Reddy, "Layerwise Fundamental Solutions and Three-Dimensional Model for Layered Media" *J. Applied Composite Materials,* Vol. 3, pp. 277–300, 1996.
- D. C. Lagoudas, C. M. D. Moorthy, and M. A., Qidwai, and J. N. Reddy, "Modeling of the Thermomechanical Response of Active Laminates with SMA Strips Using the Layerwise Finite Element Method", *J. Intelligent Mater. Syst. and Struct.,* Vol. 8, pp. 476-488, 1997.
- C. M. D. Moorthy, "Modeling Laminates Using a Layerwise Finite Element with Enhanced Strains for Interlaminar Stress Recovery and Delamination Characteristics", *Ph. D. Dissertation*, Dept. Mech. Engng., Texas A&M University, April 1997.
- K. Y. Lam, X. Q. Peng, G. R. Liu, and J. N. Reddy, "A Finite-Element Model for Piezoelectric Composite Laminates", *Smart Materials and Structures,* Vol. 6, No.5, pp. 583-591, 1997.
- John A. Mitchell, "A High Performance Iterative Solution Procedure for Solving Problems in Structural Mechanics Using the Finite Element Method", *Ph. D. Dissertation*, Dept. Mech. Engng., Texas A&M University, May 1997.
- G. Rengarajan, R. Krishna Kumar, and J. N. Reddy, "Numerical Modelling of Martensitic Phase Transformations in Shape Memory Alloys", *International Journal of Solids & Structures,* Vol. 35, No. 14, pp. 1489–1513, 1998.
- C. M. Dakshina Moorthy and J. N. Reddy, "Modeling of Laminates Using a Layerwise Element with Enhanced Strains", *Int. J. Numer. Meth. Engng.,* Vol. 43, pp. 755-779, 1998.

- G. Rengarajan, R. Krishna Kumar, and J. N. Reddy, "Numerical Modelling of Martensitic Phase Transformations in Shape Memory Alloys", *International Journal of Solids & Structures,* Vol. 35, No. 14, pp. 1489–1513, 1998.
- G. Rengarajan, "On the Inelastic Behavior of Crystalline Solids", *Ph. D. Dissertation,* Dept. Mech. Engng.,, Texas A&M University, September 1998.
- J. N. Reddy, "On Laminated Composite Plates with Integrated Sensors and Actuators", *Engineering Structures,* Vol. 21, pp. 568–593, 1999.
- Grama N. Praveen,"Modeling Inelasticity in Materials with Application to Superplasticity", *Ph. D. Dissertation,* Dept. Mech. Engng., Texas A&M University, February 1999.
- J. A. Mitchell and J. N. Reddy, "A Hierarchical Iterative Procedure for the Analysis of Composite Laminates", *Comput. Meth. Appl. Mech. Engng.,* Vol. 181, pp. 237–260, 2000.
- J. N. Reddy and J. I. Barbosa, "On Vibration Suppression of Magnetostrictive Beams", *Smart Materials and Structures,* to appear.
- J. N. Reddy and Z. Q. Cheng, "Three-Dimensional Solution of Smart Functionally Graded Plates", *Journal of Applied Mechanics,* to appear.

Books or Book Chapters Credited to ARO Grant:

- *Mechanics of Laminated Composite Plates: Theory and Analysis,* J. N. Reddy, CRC Press, Boca Raton, FL, 1997.
- *Theory and Analysis of Elastic Plates,* J. N. Reddy, Taylor & Francis, Philadelphia, PA, 1999.

Award Information: All awards received by *J. N. Reddy*(PI)

- The *Nathan M. Newmark Medal* from the American Society of Civil Engineers, 1998.
- *Fellow* of the International Association of Computational Mechanics (IACM), 1998.
- The *Melvin R. Lohmann Medal* from Oklahoma State University, Stillwater, Oklahoma, April 1997.
- The *Archie Higdon Distinguished Educator Award* from the American Society of Engineering Education, June 1997.
- "Developments in Computational Structural Dynamics". A **Keynote Lecture** presented at the *Sixth International Conference on Recent Advances in Structural Dynamics,* The Institute of Sound and Vibration Research, University of Southampton, England, July, 14–17 1997.
- "Recent Developments in Mechanics of Smart Structures". An **Opening Lecture** presented at the *Symposium on Mechanics of Composite Materials,* Instituto Superior Técnico (IST), Lisbon, Portugal, July 22, 1997.

- "An Overview and Recent Developments in Vibrations of Laminated Composite Plates and Shells", **Keynote Lecture** of the *Asia-Pacific Vibration Conference '99 (A-PVC'99),* Nanyang Technological University, Singapore, December 12-14, 1999.
- "Future Directions in Computational Methods and Simulations", **Keynote Lecture** of the *Fourth Asia-Pacific Conference on Computational Mechanics (APCOM'99),* National University of Singapore, Singapore, December 14-16, 1999.

Degrees Conferred:

- C. M. Dakshina Moorthy, "Modelling Laminates Using a Layerwise Finite Element with Enhanced Strains for Interlaminar Stress Recovery and Delamination Characteristics", *Ph.D. Dissertation*, Texas A&M University, April 1997 (working for ZENTECH, Houston, Texas).
- John A. Mitchell, "A High Performance Iterative Solution Procedure for Solving Problems in Structural Mechanics Using the Finite Element Method", *Ph.D. Dissertation*, Texas A&M University, May 1997 (working for Sandia National Laboratories, Albuquerque, New Mexico)
- Govind Rengarajan, "On the Inelastic Behavior of Crystalline Solids", *Ph.D. Dissertation*, Texas A&M University, September 1998 (working for General Electric, Schenectady, New York).
- Grama N. Praveen, "Modeling Inelasticity in Materials with Application to Superplasticity", *Ph.D. Dissertation*, Texas A&M University, December 1999 (working with Cummings Engines, Columbus, Indiana).

Technical Discussion

1. Background

In the last two decades, the subject area of *smart/intelligent materials and structures* has experienced tremondous growth in terms of research and development. Numerous conferences, workshops, and journals dedicated to smart materials and structures stand testimony to this growth. The vastness of the literature is obvious considering the interdisciplinary nature of the subject. Physicists, mathematicians, and engineers from aerospace, chemical, civil, electrical, materials, and mechanical engineering fields are all involved in some part of the development of smart materials and structural systems. One reason for this activity is that it may be possible to create certain types of structures and systems capable of adapting to or correcting for changing operating conditions. The advantage of incorporating these special types of materials into the structure is that the sensing and actuating mechanism becomes part of the structure by sensing and actuating strains directly. These types of mechanisms are referred to as *strain sensing and actuating* (SSA).

The technological implications of this class of materials are immense: structures that monitor their own health, particularly useful in remote operations, process monitoring, vibration isolation and control, and medical applications, to name only a few. On the threshold of 21st century, we are entering into research and development of the next generation of smart materials and structural systems. The next generation of smart material systems will feature *thermo-electro-mechanical coupling*, *functionality*, *intelligence*, and *miniaturization* (down to nano length scales). With the advent of these new generation of materials, reliability and integrity of these systems become central issues. These systems operate under varying conditions – they span the whole spectrum of magneto-electro-thermomechanical conditions. These conditions could vary from low to high temperatures, low to high pressures, low to high load levels, low to high strain levels, and low to high electric and magnetic fields. Such operating environments pose serious problems to the design and maintenance of the smart structural systems. Experimental investigations of both the smart material and the structural system, though possible, are prohibitively expensive, and therefore they should be complemented with theoretical analyses. The present study is concerned with one such analysis. The following review of literature provides a background for the study.

2. Review of literature

2.1 Smart materials and structures

The phrase *smart structural system* refers to a wide variety of active material and passive structural systems. For instance, a sufficiently general system is a composite (beam, plate, shell, or any other fundamental form) with embedded or surface mounted piezoelectric or electrostrictive patches, or even layers of active materials in a laminated system (see Figure 2.1). The literature contains many definitions of *smart* and *intelligent* structures. We prefer to follow Newnham's definitions[1]. The structures with surface mounted or embedded sensors and actuators, that have the capability to sense and take corrective action are referred to as smart structures. The feedback circuitry linking sensing and actuating is external to the sensor and actuator components. In fact, this precisely distinguishes a smart structural system from an intelligent structural system. Intelligent structural systems involve smart components in

which the functions of sensing, feedback control, and actuating are all integrated. This type of system finds applications in aircraft wings, helicopter rotors, weapon systems, automobiles, and so on.

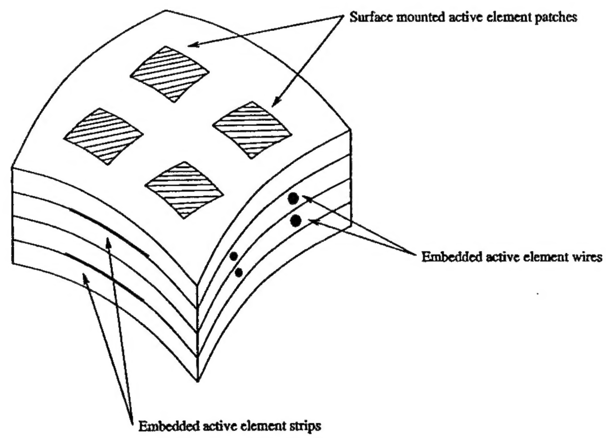

Figure 2.1: A schematic of a laminated plate with surface mounted piezoelectric patches.

Essentially, a smart structural system is a multi-functional unit (see Figure 2.2). It has a load (electrical, thermal, magnetic, or mechanical) bearing part which is usually passive, and an active material part that performs the operations of sensing and actuating. For example, vibration amplitudes in a flexible plate structure may be suppressed using the sensing and actuation capabilities of piezoelectric or piezoceramic films by bonding them to the surfaces of the plate. As the plate deforms due to external applied loads, the bonded piezoelectric film (sensor) also deforms, and due to its constitutive behavior, it develops a surface charge proportional to the applied force. The charge may be processed by a control system, which supplies an appropriate voltage to the piezolectric film (actuator) that induces a counteractive deformation to the plate structure and the amplitudes of vibrations are suppressed.

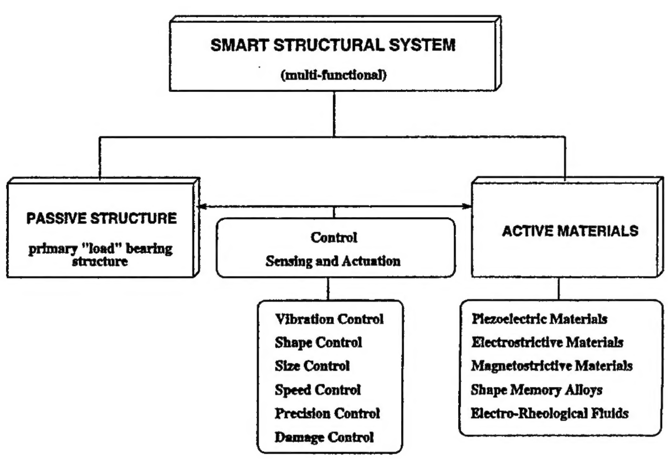

Figure 2.2: Components of a smart structural system.

Piezoelectric materials, electrostrictive materials, magnetostrictive materials, shape memory alloys, and electro-rheological fluids are some of the smart materials available today. Among the currently available sensors and actuators, the smallest ones are of the order of few millimeters. However, progress towards intelligent structures requires us to develop smart material systems that are of the order of a few microns. The reduction in size has tremendous technological benefits; however, clear understanding of reliability and system integrity are vital to the efficient and optimum use of these material systems. As dimensions get smaller, induced electro-thermo-mechanical fields get larger. Therefore, the convenience of linearity in modeling should be abandoned and material and geometric nonlinearities should be accounted for.

2.2 Piezoelectric materials

Piezoelectricity is a phenomenon in which some materials develop polarization upon application of strains[2]. This phenomenon is observed in materials that have a non-centrosymmetric crystal structure. Examples of piezoelectric materials are Rochelle salt, quartz, and the most popular one, Lead Zirconate Titanate or PZT (Pb (Zr,Ti) O_3).

Piezoelectric materials exhibit a linear relationship between the electric field and strains for low field values (up to 100 V/mm). However, the relationship is nonlinear for large fields, and the material exhibits hysteresis[3]. Furthermore, piezoelectric materials show dielectric aging and hence lack reproducibility of strains, *i.e.*, a drift from zero state of strain is observed under cyclic electric field conditions[4].

2.3 Electrostrictive materials

Electrostriction is a second-order effect observed in all dielectrics (centrosymmetric and non-centrosymmetric crystal structure) where an applied uniform electric field induces a strain, but a reversal of the field does not alter the strain[2]. That is, the strain is proportional to the square of the electric field. An electrostrictive material that is being used increasingly is Lead Magnesium Niobate or PMN (Pb ($Mg_{1/3}$ $Nb_{2/3}$) O_3). These materials have a nonlinear field-strain relationship, but exhibit very little, if any, hysteresis[5]. Additionally, these materials have excellent zero strain reproducibility in cyclic electric field conditions[5]. However, strains induced by electrostriction are comparable to piezoelectricity only at large electric fields. But the voltage can be reduced to the piezoelectric level for an electrostrictive material with a high dielectric constant[5]. Table 2.1 summarizes the characteristics of piezoelectric and electrostrictive materials.

2.4 Composites

The active composite elements are available in two forms: multi-layered active materials and multi–phase active materials. The multi-layered active materials are stacks of active materials wherein it is possible to achieve large fields for moderate voltage input levels[9–14]. A typical multilayer piezoelectric actuator is shown in Figure 2.3. The multilayer active ceramics are manufactured using tape casting technology and electrodes are embedded in several different ways[9,10]. The multi-layer composite system is also amenable to miniaturization as it is possible to decrease the thickness of active material layers. However, accounting for nonlinearity in modeling is essential. Considerable research and development has been carried out in piezoelectric multi-layered ceramics. Active vibration control using these multi-layered piezoceramic configurations has been investigated[15–17].

Table 2.1. Characteristics of Piezoelectric and Electrostrictive Materials.

Piezoelectric Materials

- At low electric fields, the strain–electric field relationship is linear.
- At large electric fields, the relationship is nonlinear, and hysteresis appears under cyclic field conditions.
- Exhibits drift from zero strain state under cyclic field conditions due to dielectric aging.
- In the linear regime, the piezoelectric coefficient is constant and hence cannot be electrically tuned with a bias field.
- Has been studied extensively in the last few years, and several analysis tools have emerged; experimental characterization has been done extensively.
- Currently still popular because of lower power requirements at high frequencies, relatively lower prices, and above all *an understanding of the component integrity at low fields.*

Electrostrictive Materials

- The strain–electric field relationship is nonlinear.
- Shows very little, if any, hysteresis.
- *Has excellent reproducibility of zero strain state.*
- Requires a high dielectric constant to generate a large electromechanical coupling.
- Nonlinear relation between the strain and field can be used to tune the dielectric constant; therefore, the electromechanical coefficient can be tuned over a wide range, changing from inactive to extremely active states.
- While the constitutive relationship has been in existence in the literature for quite a while[5–8], modeling and analysis of the behavior of electrostrictive materials under various loading conditions have not been carried out. Further understanding of reliability and structural integrity of smart structural systems with electrostrictive materials is necessary.

Multi-phase composite systems consist of an active phase embedded in a passive matrix (for example, PZT rods in a polymer matrix[18–20], as shown in Figure 2.4). Different connectivity schemes are possible in a two-phase system. The classification based on connectivity of active and passive phases was developed by Newnham, *et al.*[21]. Literature abounds with papers on piezoceramic rods (active) embedded in polymer matrix (passive), forming a 1-3 connectivity composite system[18,19]. Finite element analysis of 1-3 composite systems has also been carried out[22,23]. These multi–phase composites are most often useful

when there are conflicting requirements on the active material system. This type of active–passive phase combinations helps in achieving multi-functionality. However, for reliability, mechanics studies relating to active/passive material bonding interactions are necessary[24]. Certain receptor cells in the cochlea of the inner ear undergo displacements five orders of magnitude more than what the best piezoelectric ceramic can produce at the same level of applied voltage and dimensionality[25].

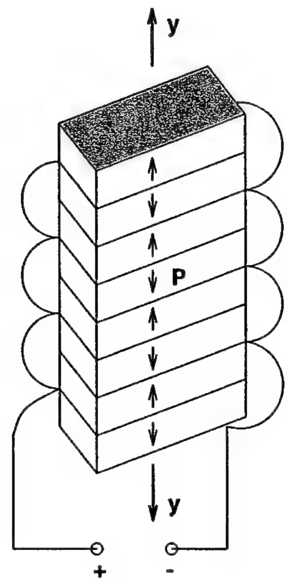

Figure 2.3: A multi-layer actuator.

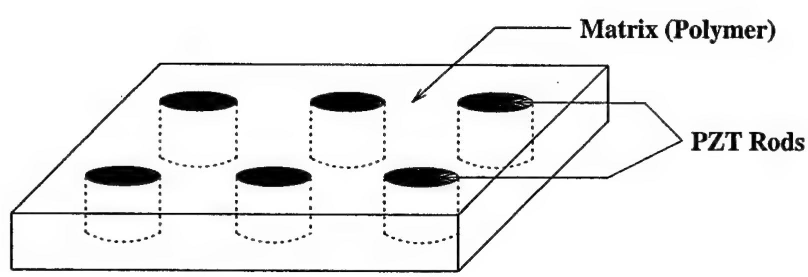

Figure 2.4: A schematic of a 1-3 connectivity composite.

2.5 Issues in modeling and analysis

The mechanics of smart material systems involves coupling between electric, magnetic, thermal, and mechanical effects. In addition to this coupling, it may be necessary to account for geometric and material nonlinearities. For example, an electromechanical transducer is characterized by five important properties[1]: the resonant frequency, acoustic impedance,

mechanical damping coefficient, electromechanical coupling coefficient, and the electric impedance. If nonlinear electroelastic equations are included in the model, some or all of these properties can be tuned; for instance, in an electrostrictive material, the electromechanical coupling coefficient can be tuned with a bias field[1]. In order to tune the first fundamental resonant frequency of the transducer, thin rubber layers are introduced in a multilayer PZT laminate[26]. The thin rubber layers necessitate the use of nonlinear elastic relations.

Toupin[27] was the first one to consider the nonlinearity in electroelastic formulations. Knops[28] presented a two-dimensional theory of electrostriction and solved a simplified boundary value problem using complex potentials. However, rotationally invariant nonlinear thermoelectroelastic equations were derived by Tiersten and his coworkers[6,7,9], and by Nelson[8]. Tiersten[9] has stressed the importance of including nonlinear terms in the constitutive relations, particularly at large fields. Joshi[29] has also presented nonlinear constitutive relations for piezoceramic materials.

To our knowledge, a fully coupled, nonlinear electrothermomechanical analysis has not been carried out so far. Analytical studies have largely concentrated on fracture of piezoceramics and electrostrictive ceramics. Cracks emanating from electrode junctions, debonding in active/passive composite systems, and delamination in multi-layered stacking are some of the analytical problems that have been considered in the literature[30,31]. Coupled analyses were carried out for linear piezoelectric materials[32].

During the present research, a general formulation for laminated composite plates with piezoelectric/magnetostrictive actuators and/or sensors has been developed. Formulations of both classical and shear deformation (first-order and third-order) theories were developed, and they take into account the thermo-electro-mechanical coupling, von Kármán type geometric nonlinearity, and time dependency. The Navier solutions for the linear cases were also derived. Assumed kinematics of deformation through thickness of the plate in various plate theories is graphically depicted in Figure 2.5.

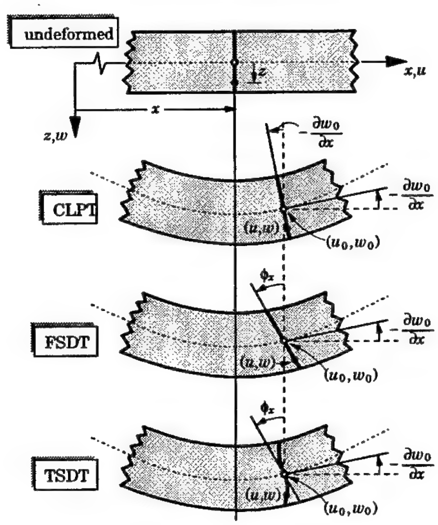

Figure 2.5: Kinematics of deformation of a plate edge in various plate theories.

3. Theoretical formulation using CLPT

3.1 Displacements and strains

The *classical laminated plate theory* (CLPT) is based on the Kirchhoff assumption (see Reddy[51–53]) that straight lines perpendicular to the midsurface (*i.e.*, transverse normals) before deformation remain straight after deformation, they are inextensible, and rotate such that they remain perpendicular to the midsurface after deformation. These assumptions imply that the transverse normal strain ε_{zz} and transverse shear strains ε_{xz} and ε_{yz} are zero.

Consider a plate of total thickness h composed of N orthotropic layers with the principal material coordinates of the kth lamina oriented at an angle θ_k to the laminate coordinate, x. The $xy-$plane is taken to be the undeformed midplane Ω_0 of the laminate, and the $z-$axis is taken positive upward from the midplane. The kth layer is located between the points $z = z_k$ and $z = z_{k+1}$ in the thickness direction. Some of the layers have the sole purpose of actuating or sensing deformation of the laminate (see Figure 3.1).

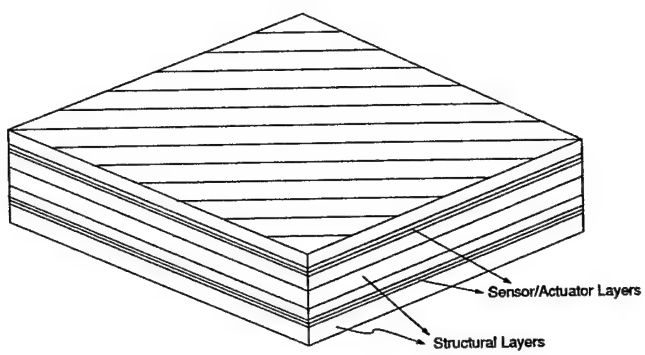

Figure 3.1: A schematic of a laminated plate with imbedded actuating/sensing layers.

In formulating the theory, we assume that the layers are perfectly bonded together. Further, restrict the formulation to linear elastic material bahavior, small strains and displacements, and to the case in which the temperature and electric fields are given.

The Kirchhoff hypothesis leads to the displacement field

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) - z\frac{\partial w_0}{\partial x} \\
v(x,y,z,t) &= v_0(x,y,t) - z\frac{\partial w_0}{\partial y} \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
\tag{3.1}
$$

where (u_0, v_0, w_0) are the displacements along the coordinate lines of a material point on the $xy-$plane. Once the midplane displacements (u_0, v_0, w_0) are known, the displacements of any arbitrary point (x, y, z) in the 3-D continuum can be determined using Eq. (3.1).

The nonzero von Kármán strains associated with the displacement field in Eq. (3.1) are given by

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} \tag{3.2a}$$

$$\{\varepsilon^0\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\end{array}\right\}, \quad \{\varepsilon^1\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} = \left\{\begin{array}{c}-\frac{\partial^2 w_0}{\partial x^2}\\ -\frac{\partial^2 w_0}{\partial y^2}\\ -2\frac{\partial^2 w_0}{\partial x \partial y}\end{array}\right\} \tag{3.2b}$$

where $(\varepsilon_{xx}^{(0)}, \varepsilon_{yy}^{(0)}, \gamma_{xy}^{(0)})$ are the *membrane strains* and $(\varepsilon_{xx}^{(1)}, \varepsilon_{yy}^{(1)}, \gamma_{xy}^{(1)})$ are the flexural (bending) strains. The transverse strains $(\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{zz})$ are identically zero in the classical plate theory. Note from Eq. (3.2b) that all strain components vary linearly through the laminate thickness, and they are independent of the material variations through the laminate thickness.

3.2 Lamina constitutive relations

The linear constitutive relations for the kth orthotropic (piezoelectric) lamina in the principal material coordinates of a lamina are

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \sigma_6\end{array}\right\}^{(k)} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\varepsilon_1 - \alpha_1\,\Delta T\\ \varepsilon_2 - \alpha_2\,\Delta T\\ \varepsilon_6\end{array}\right\} - \begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{32}\\ 0 & 0 & 0\end{bmatrix}^{(k)} \left\{\begin{array}{c}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{array}\right\}^{(k)} \tag{3.3}$$

where $Q_{ij}^{(k)}$ are the plane stress-reduced stiffnesses and $e_{ij}^{(k)}$ are the piezoelectric moduli of the kth lamina, $(\sigma_i, \varepsilon_i, \mathcal{E}_i)$ are the stress, strain, and electric field components, respectively, referred to the material coordinate system (x_1, x_2, x_3), α_1 and α_2 are the coefficients of thermal expansion along the x_1 and x_2 directions, respectively, and ΔT is the temperature increment from a reference state, $\Delta T = T - T_0$. For layers other than piezoelectric layers, the part containing the piezoelectric moduli $e_{ij}^{(k)}$ should be omitted. The coefficients $Q_{ij}^{(k)}$ are known in terms of the engineering constants of the kth layer:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12} \tag{3.4a}$$

and the piezoelectric stiffnesses are known in terms of the dielectric constants and elastic stiffnesses as

$$\begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{32}\\ 0 & 0 & 0\end{bmatrix}^{(k)} = \begin{bmatrix}0 & 0 & d_{31}\\ 0 & 0 & d_{32}\\ 0 & 0 & 0\end{bmatrix}^{(k)} \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}^{(k)} \tag{3.4b}$$

Since the laminate is made of several orthotropic layers, with their material axes oriented arbitrarily with respect to the laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates (x, y, z). The transformed stress–strain relations relate the stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ to the strains $(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy})$ and components of the electric displacement vector $(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z)$ in the laminate coordinates

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}^{(k)} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^{(k)} \left(\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} - \left\{\begin{array}{c}\bar{\alpha}_1\\ \bar{\alpha}_2\\ \bar{\alpha}_6\end{array}\right\}^{(k)} \Delta T\right) - \begin{bmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{32}\\ 0 & 0 & \bar{e}_{36}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{array}\right\}^{(k)} \tag{3.5}$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned} \tag{3.6}$$

and $\bar{\alpha}_1$, $\bar{\alpha}_2$, and $\bar{\alpha}_6$ are the transformed thermal coefficients of expansion

$$\begin{aligned}
\bar{\alpha}_1 &= \alpha_1\cos^2\theta + \alpha_2\sin^2\theta \\
\bar{\alpha}_2 &= \alpha_1\sin^2\theta + \alpha_2\cos^2\theta \\
\bar{\alpha}_6 &= 2(\alpha_1 - \alpha_2)\sin\theta\cos\theta
\end{aligned} \tag{3.7}$$

and $\bar{e}_{ij}$ are the transformed piezoelectric moduli

$$\begin{aligned}
\bar{e}_{31} &= e_{31}\cos^2\theta + e_{32}\sin^2\theta = Q_{66}\left(d_{31}\cos^2\theta + d_{32}\sin^2\theta\right) \\
\bar{e}_{32} &= e_{31}\sin^2\theta + e_{32}\cos^2\theta = Q_{66}\left(d_{31}\sin^2\theta + d_{32}\cos^2\theta\right) \\
\bar{e}_{36} &= (e_{31} - e_{32})\sin\theta\cos\theta = Q_{66}(d_{31} - d_{32})\sin\theta\cos\theta
\end{aligned} \tag{3.8}$$

Here θ is the angle measured counterclockwise from the $x-$coordinate to the x_1-coordinate. We define

$$\{\bar{e}\}^{(k)} = \begin{Bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_6 \end{Bmatrix}^{(k)} \equiv \begin{Bmatrix} \bar{e}_{31} \\ \bar{e}_{32} \\ \bar{e}_{36} \end{Bmatrix}^{(k)} \tag{3.9}$$

where summation is implied on repeated subscript j over the range $j = 1, 2, 6$.

3.3 Equations of motion

The equations of motion can be derived using the principle of virtual displacements. In the derivations, we account for thermal (and hence, moisture) and piezoelectric effects only with the understanding that the material properties are independent of temperature and electric fields, and that the temperature T and electric displacement vector $\mathcal{E}$ are known functions of position. Thus temperature and electric fields enter the formulation only through constitutive equations.

The equations of motion of the classical plate theory are

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \tag{3.10}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} - I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right) \tag{3.11}$$

$$\begin{aligned}
&\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial y\partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + \mathcal{N}(w_0, N_{xx}, N_{xy}, N_{yy}) + q \\
&\qquad = I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right)
\end{aligned} \tag{3.12}$$

where q is the distributed transverse mechanical load, and (N_{xx}, N_{yy}, N_{xy}) the total in-plane *force resultants* and (M_{xx}, M_{yy}, M_{xy}) the total *moment resultants* defined by

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz \tag{3.13}$$

(I_0, I_1, I_2) are the mass moments of inertia

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \rho_0\, dz \tag{3.14}$$

and

$$\mathcal{N} = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) \tag{3.15}$$

Note that the force and moment resultants are related to the mechanical, thermal, and piezoelectric strains through the constitutive equations. We will consider the laminate constitutive equations in the sequel.

The generalized displacements and forces of CLPT are

$$u_n, \quad u_s, \quad w_0, \quad \frac{\partial w_0}{\partial n} \quad \text{(essential)}$$

$$N_{nn}, \quad N_{ns}, \quad V_n \equiv Q_n + \frac{\partial M_{ns}}{\partial s}, \quad M_{nn} \quad \text{(natural)} \tag{3.16}$$

where Q_n denotes the transverse shear force resultant on an edge with normal $\hat{\mathbf{n}} = (n_x, n_y)$. In the dynamic case, it is given by

$$\begin{aligned} Q_n \equiv & \left(M_{xx,x} + M_{xy,y} - I_1\ddot{u}_0 + I_2\frac{\partial \ddot{w}_0}{\partial x}\right) n_x + \left(M_{yy,y} + M_{xy,x} - I_1\ddot{v}_0 + I_2\frac{\partial \ddot{w}_0}{\partial y}\right) n_y \\ & + \left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) n_x + \left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) n_y \end{aligned} \tag{3.17}$$

The force and moment resultants on an edge with normal $\hat{\mathbf{n}}$ to those on edges parallel to the x and y coordinates by

$$\begin{Bmatrix} N_{nn} \\ N_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} \tag{3.18a}$$

$$\begin{Bmatrix} M_{nn} \\ M_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \tag{3.18b}$$

3.4 Laminate constitutive equations

Here we relate the force and moment resultants in Eq. (3.13) to the strains. By definition, we have

$$
\begin{aligned}
\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \end{array}\right\} &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right\} dz \\
&= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left(\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \left(\left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array}\right\} - \left\{\begin{array}{c} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_6 \end{array}\right\}^{(k)} \Delta T \right) - \left\{\begin{array}{c} \bar{e}_{31} \\ \bar{e}_{32} \\ \bar{e}_{36} \end{array}\right\}^{(k)} \mathcal{E}_z^{(k)} \right) \\
&= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array}\right\} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array}\right\} - \left\{\begin{array}{c} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{array}\right\} - \left\{\begin{array}{c} N_{xx}^P \\ N_{yy}^P \\ N_{xy}^P \end{array}\right\}
\end{aligned}
\tag{3.19}
$$

where $\{N^T\}$ and $\{N^P\}$ are thermal and electric force resultants

$$\{N^T\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\bar{\beta}\}^{(k)} \Delta T \, dz \tag{3.20}$$

$$\{N^P\} = \sum_{k=1}^{N_a} \int_{z_k}^{z_{k+1}} \{\bar{e}\}^{(k)} \mathcal{E}_z^{(k)} \, dz \tag{3.21}$$

N_a is the number of actuating layers, and

$$\{\bar{\beta}\}^{(k)} = [\bar{Q}]^{(k)} \{\bar{\alpha}\}^{(k)} \tag{3.22}$$

Similarly, we have

$$
\left\{\begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right\} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array}\right\} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array}\right\} - \left\{\begin{array}{c} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{array}\right\} - \left\{\begin{array}{c} M_{xx}^P \\ M_{yy}^P \\ M_{xy}^P \end{array}\right\}
\tag{3.23}
$$

where $\{M^T\}$ and $\{M^P\}$ are thermal and electric moment resultants

$$\{M^T\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\bar{\beta}\}^{(k)} \Delta T \; z \, dz \tag{3.24}$$

$$\{M^P\} = \sum_{k=1}^{N_a} \int_{z_k}^{z_{k+1}} \{\bar{e}\}^{(k)} \mathcal{E}_z^{(k)} \; z \, dz \tag{3.25}$$

Here A_{ij} denote the *extensional stiffnesses*, D_{ij} the *bending stiffnesses*, and B_{ij} the *bending–extensional coupling stiffnesses*

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (z_{k+1} - z_k) \; , \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3) \tag{3.26}$$

Note that $\bar{Q}$'s, and therefore A's, B's, and D's, are, in general, functions of position (x, y). Equations (3.19) and (3.23) can be written in a compact form as

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\varepsilon^1\} \end{Bmatrix} - \begin{Bmatrix} \{N^T\} \\ \{M^T\} \end{Bmatrix} - \begin{Bmatrix} \{N^P\} \\ \{M^P\} \end{Bmatrix} \tag{3.27}$$

Suppose that the temperature and electric fields vary linearly within kth layer

$$\Delta T = T_1^k(x, y, t)\psi_1^k(z) + T_2^k(x, y, t)\psi_2^k(z) \tag{3.28a}$$

$$\mathcal{E}_z = \mathcal{E}_1^k(x, y, t)\psi_1^k(z) + \mathcal{E}_2^k(x, y, t)\psi_2^k(z) \tag{3.28b}$$

where ψ_i^k are the linear interpolation functions of the kth layer

$$\psi_1^k(z) = \frac{z_{k+1} - z}{h_k}, \quad \psi_2^k(z) = \frac{z - z_k}{h_k}, \quad z_k \le z \le z_{k+1} \tag{3.29}$$

and T_1^k and T_2^k denote the temperature at the bottom and top surfaces of the kth layer. Similar notation is used for the electric field. Then the thermal and electrical forces and moments can be evaluated as

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_6 \end{Bmatrix}^{(k)} (T_1^k\psi_1^k + T_2^k\psi_2^k)dz \equiv \begin{Bmatrix} A_1^T \\ A_2^T \\ A_6^T \end{Bmatrix} \tag{3.30}$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_6 \end{Bmatrix}^{(k)} (T_1^k\psi_1^k + T_2^k\psi_2^k)zdz \equiv \begin{Bmatrix} B_1^T \\ B_2^T \\ B_6^T \end{Bmatrix} \tag{3.31}$$

$$\begin{Bmatrix} N_{xx}^P \\ N_{yy}^P \\ N_{xy}^P \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{d}_{31} \\ \bar{d}_{32} \\ \bar{d}_{36} \end{Bmatrix}^{(k)} (\mathcal{E}_1^k\psi_1^k + \mathcal{E}_2^k\psi_2^k)dz \equiv \begin{Bmatrix} A_1^P \\ A_2^P \\ A_6^P \end{Bmatrix} \tag{3.32}$$

$$\begin{Bmatrix} M_{xx}^P \\ M_{yy}^P \\ M_{xy}^P \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{d}_{31} \\ \bar{d}_{32} \\ \bar{d}_{36} \end{Bmatrix}^{(k)} (\mathcal{E}_1^k\psi_1^k + \mathcal{E}_2^k\psi_2^k)zdz \equiv \begin{Bmatrix} B_1^P \\ B_2^P \\ B_6^P \end{Bmatrix} \tag{3.33}$$

where $(i = 1, 2, 6)$

$$A_i^T = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1,2,6} \bar{Q}_{ij}^{(k)}\bar{\alpha}_j^{(k)}\left(T_1^k + T_2^k\right)h_k \tag{3.34a}$$

$$B_i^T = \frac{1}{6}\sum_{k=1}^{N}\sum_{j=1,2,6} \bar{Q}_{ij}^{(k)}\bar{\alpha}_j^{(k)}\left[T_1^k\left(h_k + 3z_k\right) + T_2^k\left(2h_k + 3z_k\right)\right]h_k \tag{3.34b}$$

are the thermal stiffnesses, and

$$A_i^P = \frac{1}{2}\sum_{k=1}^{N_a}\sum_{j=1,2,6} \bar{Q}_{ij}^{(k)}\bar{d}_{3j}^{(k)}\left(\mathcal{E}_1^k + \mathcal{E}_2^k\right)h_k \tag{3.35a}$$

$$B_i^P = \frac{1}{6}\sum_{k=1}^{N_a}\sum_{j=1,2,6} \bar{Q}_{ij}^{(k)}\bar{d}_{3j}^{(k)}\left[\mathcal{E}_1^k\left(h_k + 3z_k\right) + \mathcal{E}_2^k\left(2h_k + 3z_k\right)\right]h_k \tag{3.35b}$$

are the piezoelectric stiffnesses, and $N_a < N$ is the number of actuator layers. If the electric field intensity is constant across each lamina, then one may approximate it as

$$\mathcal{E}_1^k = \mathcal{E}_2^k = \frac{V_k}{h_k} \tag{3.36}$$

where V_k is the applied voltage across the kth layer and h_k is the thickness of the layer.

The stress resultants (N's and M's) are related to the displacement gradients, temperature increment, and electric field by the relations

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$
$$- \begin{Bmatrix} A_1^T \\ A_2^T \\ A_6^T \end{Bmatrix} - \begin{Bmatrix} A_1^P \\ A_2^P \\ A_6^P \end{Bmatrix} \tag{3.37}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$
$$- \begin{Bmatrix} B_1^T \\ B_2^T \\ B_6^T \end{Bmatrix} - \begin{Bmatrix} B_1^P \\ B_2^P \\ B_6^P \end{Bmatrix} \tag{3.38}$$

4. Theoretical formulation of TSDT

4.1. General comments

The classical laminate plate theory and the first-order shear deformation theory are the simplest equivalent single-layer theories, and they adequately describe the kinematic behavior of most laminates[52]. The third-order theory of Reddy[51,52,54] represents the plate kinematics better, does not require shear correction factors, and yields more accurate interlaminar stress distributions. Hence, the generalized third-order shear deformation laminate theory (TSDT) of Reddy[52], which contains the first-order shear deformation theory (FSDT) as a special case, is used to develop the governing equations of laminated plates with actuating and/or sensing layers.

4.2. Displacement field and strains

The Reddy third-order plate theory is based on the displacement field

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) - c_1 z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) - c_1 z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{4.1}$$

where (u_0, v_0, w_0) and (ϕ_x, ϕ_y) have the same physical meaning as in the first-order theory; they denote the displacements and rotations of transverse normals on the plane $z = 0$, respectively. Then the displacement field of FSDT is obtained by setting $c_1 = 0$, and for the Reddy third-order theory we set $c_1 = 4/(3h^2)$.

Substitution of the displacements (4.1) into the von Kármán nonlinear strain–displacement relations yields the strains[52]

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} + z^3\left\{\begin{array}{c}\varepsilon_{xx}^{(3)}\\ \varepsilon_{yy}^{(3)}\\ \gamma_{xy}^{(3)}\end{array}\right\} \tag{4.2}$$

$$\left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\} = \left\{\begin{array}{c}\gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\end{array}\right\} + z^2\left\{\begin{array}{c}\gamma_{yz}^{(2)}\\ \gamma_{xz}^{(2)}\end{array}\right\} \tag{4.3}$$

where

$$\left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\end{array}\right\}, \quad \left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial \phi_x}{\partial x}\\ \frac{\partial \phi_y}{\partial y}\\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\end{array}\right\} \tag{4.4}$$

$$\left\{\begin{array}{c}\varepsilon_{xx}^{(3)}\\ \varepsilon_{yy}^{(3)}\\ \gamma_{xy}^{(3)}\end{array}\right\} = -c_1\left\{\begin{array}{c}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right)\\ \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right)\\ \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y}\right)\end{array}\right\} \tag{4.5}$$

$$\left\{\begin{array}{c}\gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\end{array}\right\} = \left\{\begin{array}{c}\phi_y + \frac{\partial w_0}{\partial y}\\ \phi_x + \frac{\partial w_0}{\partial x}\end{array}\right\}, \quad \left\{\begin{array}{c}\gamma_{yz}^{(2)}\\ \gamma_{xz}^{(2)}\end{array}\right\} = -3c_1\left\{\begin{array}{c}\left(\phi_y + \frac{\partial w_0}{\partial y}\right)\\ \left(\phi_x + \frac{\partial w_0}{\partial x}\right)\end{array}\right\} \tag{4.6}$$

4.3. Constitutive relations

In addition to the constitutive relations in Eq. (3.5), we have the following requations for the transverse shear stresses

$$\left\{\begin{array}{c}\sigma_{yz}\\ \sigma_{xz}\end{array}\right\} = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}\left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\} + \begin{bmatrix}\bar{e}_{14} & \bar{e}_{24} & 0\\ \bar{e}_{15} & \bar{e}_{25} & 0\end{bmatrix}\left\{\begin{array}{c}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{array}\right\} \tag{4.7}$$

where

$$\begin{aligned}\bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta\\ \bar{Q}_{45} &= (Q_{55} - Q_{44})\cos\theta\sin\theta\\ \bar{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta\end{aligned} \tag{4.8}$$

$$\begin{aligned}\bar{e}_{14} &= (e_{15} - e_{24})\sin\theta\cos\theta\\ \bar{e}_{24} &= e_{24}\cos^2\theta + e_{15}\sin^2\theta\\ \bar{e}_{15} &= e_{15}\cos^2\theta + e_{24}\sin^2\theta\end{aligned}$$

$$\bar{e}_{25} = (e_{15} - e_{24}) \sin\theta \cos\theta \tag{4.9}$$

$$\begin{aligned}
\bar{\epsilon}_{11} &= \epsilon_{11} \cos^2\theta + \epsilon_{22} \sin^2\theta \\
\bar{\epsilon}_{22} &= \epsilon_{11} \sin^2\theta + \epsilon_{22} \cos^2\theta \\
\bar{\epsilon}_{12} &= (\epsilon_{11} - \epsilon_{22}) \sin\theta \cos\theta
\end{aligned} \tag{4.10}$$

Also, we have

$$\begin{bmatrix} \bar{e}_{14} & \bar{e}_{15} \\ \bar{e}_{24} & \bar{e}_{25} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{d}_{14} & \bar{d}_{15} \\ \bar{d}_{24} & \bar{d}_{25} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \tag{4.11}$$

4.4. Equations of motion

The equations of motion may be developed using the dynamic version of the principle of virtual displacements.

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 + J_1 \ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x} \tag{4.12}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v}_0 + J_1 \ddot{\phi}_y - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y} \tag{4.13}$$

$$\begin{aligned}
&\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + \frac{\partial}{\partial x}(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}) + \frac{\partial}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}) \\
&+ c_1(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}) + q = I_0 \ddot{w}_0 - c_1^2 I_6 (\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}) \\
&+ c_1 \left[I_3 (\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}) + J_4 (\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}) \right]
\end{aligned} \tag{4.14}$$

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = J_1 \ddot{u}_0 + K_2 \ddot{\phi}_x - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial x} \tag{4.15}$$

$$\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = J_1 \ddot{v}_0 + K_2 \ddot{\phi}_y - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial y} \tag{4.16}$$

where

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta}, \quad \bar{Q}_\alpha = Q_\alpha - 3c_1 R_\alpha \tag{4.17}$$

$$I_i = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \rho^{(k)} (z)^i \, dz \quad (i = 0, 1, 2, \cdots, 6) \tag{4.18a}$$

$$J_i = I_i - c_1 I_{i+2}, \; K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6, \; c_1 = \frac{4}{3h^2} \tag{4.18b}$$

and (P_{xx}, P_{yy}, P_{xy}) and (R_x, R_y) denote the higher-order stress resultants

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z^3 dz, \quad \begin{Bmatrix} R_x \\ R_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} z^2 dz \tag{4.19}$$

The primary and secondary variables of the theory are

$$\text{Primary Variables}: \quad u_n,\ u_s,\ w_0,\ \frac{\partial w_0}{\partial n},\ \phi_n,\ \phi_s \tag{4.20a}$$

$$\text{Secondary Variables}: \quad N_{nn},\ N_{ns},\ \bar{V}_n,\ P_{nn},\ \bar{M}_{nn},\ \bar{M}_{ns} \tag{4.20b}$$

where P_{nn} has the form similar to M_{nn} [see Eqs. (2.18a,b)] and

$$\begin{aligned}
\bar{V}_n \equiv\ & c_1 \left[\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) n_x + \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} \right) n_y \right] \\
& - c_1 \left[\left(I_3 \ddot{u}_0 + J_4 \ddot{\phi}_x - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial x} \right) n_x + \left(I_3 \ddot{v}_0 + J_4 \ddot{\phi}_y - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial y} \right) n_y \right] \\
& + (\bar{Q}_x n_x + \bar{Q}_y n_y) + \mathcal{P}(w_0, N_{xx}, N_{yy}, N_{xy}) + c_1 \frac{\partial P_{ns}}{\partial s}
\end{aligned} \tag{4.21}$$

$$\mathcal{P} = \left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left(N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) n_y \tag{4.22}$$

The stress resultants are related to the strains by the relations

$$\left\{ \begin{matrix} \{N\} \\ \{M\} \\ \{P\} \end{matrix} \right\} = \begin{bmatrix} [A] & [B] & [E] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} \left\{ \begin{matrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(3)}\} \end{matrix} \right\} - \left\{ \begin{matrix} \{N^T\} \\ \{M^T\} \\ \{P^T\} \end{matrix} \right\} - \left\{ \begin{matrix} \{N^P\} \\ \{M^P\} \\ \{P^P\} \end{matrix} \right\} \tag{4.23}$$

$$\left\{ \begin{matrix} \{Q\} \\ \{R\} \end{matrix} \right\} = \begin{bmatrix} [A] & [D] \\ [D] & [F] \end{bmatrix} \left\{ \begin{matrix} \{\gamma^{(0)}\} \\ \{\gamma^{(2)}\} \end{matrix} \right\} \tag{4.24}$$

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}\ (1, z, z^2, z^3, z^4, z^6)\ dz \tag{4.25a}$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}\ (1, z^2, z^6)\ dz \tag{4.25b}$$

The stiffnesses A_{ij}, D_{ij} and F_{ij} are defined for $i,j = 1,2,6$ as well as $i,j = 4,5$. The stiffnesses B_{ij}, E_{ij} and H_{ij} are defined only for $i,j = 1,2,6$. The coefficients A_{ij}, B_{ij}, and D_{ij} were given in terms of the layer stiffnesses $\bar{Q}_{ij}^{(k)}$ and layer coordinates z_{k+1} and z_k in Eq. (2.26). The higher-order stiffness coefficients are defined by

$$\begin{aligned}
E_{ij} &= \frac{1}{4} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^4 - (z_k)^4 \right] \\
F_{ij} &= \frac{1}{5} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^5 - (z_k)^5 \right] \\
H_{ij} &= \frac{1}{7} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^7 - (z_k)^7 \right]
\end{aligned} \tag{4.26}$$

Note that the stiffnesses E_{ij}, F_{ij} and so on of the third-order theory involve fourth or higher powers of the thickness, and, therefore, they are expected to contribute little to thin laminate solutions. Even for moderately thick laminates the contribution can be small.

5. Sensor and actuator equations

5.1. Electro-mechanical coupling

The electro-mechanical coupling is a two-way coupling. The effect of applied electrical field on stresses is, as indicated by Eq. (2.5), is analogous to the effect of the temperature field. The components D_i of electrical displacement vector are related to the components of strains and electrical field by

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} & e_{32} & 0 \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}^{(k)} \begin{Bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{Bmatrix}^{(k)} \tag{5.1}$$

where e_{ij} are piezoelectric stiffnesses and ϵ_{ij} are dielectric coefficients. The transformed equations are

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix}^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{e}_{31} & \bar{e}_{32} & \bar{e}_{36} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix}^{(k)} \begin{Bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{Bmatrix}^{(k)} \tag{5.2a}$$

Equation (5.2a) is modified to account for the transverse shear strains in TSDT

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{e}_{31} & \bar{e}_{32} & \bar{e}_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + \begin{bmatrix} \bar{e}_{14} & \bar{e}_{15} \\ \bar{e}_{24} & \bar{e}_{25} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} - \begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \begin{Bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{Bmatrix} \tag{5.2b}$$

This electromechanical equation provides the starting point for the derivation of the sensor and actuator equations[12,13,15].

5.2. Sensor equations

According to the Gauss law, the closed circuit charge measured through the electrodes of the kth layer is

$$Q_s^k(t) = \frac{1}{2}\left[\int_{S_I(z=z_k)} D_z dxdy + \int_{S_I(z=z_{k+1})} D_z dxdy\right] \tag{5.3}$$

where $S_I = S_b \cap S_t$ is the intersection of the electrode surfaces on both sides of the lamina (see Figure 5.1) and subscript 's' denotes sensor. The electrodes are assumed to be placed on the transverse surfaces with the poling direction z. Hence, only the component D_z of the electric displacement vector is nonzero. When Eq. (5.2) is applied to sensors where the converse piezoelectric effect is negligible and the external applied charge is zero, the sensed charge can be calculated from Eq. (5.2) as

$$D_z = \{\bar{e}_{31} \;\; \bar{e}_{32} \;\; \bar{e}_{36}\} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{5.4}$$

The total charge in the laminate is calculated by summing over the number of sensor layers, N_s:

$$Q_s^k(t) = \frac{1}{2}\sum_{k=1}^{N_s}\left[\int_{S_I(z=z_k)} D_z dxdy + \int_{S_I(z=z_{k+1})} D_z dxdy\right] \tag{5.5}$$

Substituting Eq. (5.4) into (5.5), we obtain

$$Q_s(t) = \frac{1}{2}\sum_{k=1}^{N_s}\int_{S_I}\left[\bar{e}_{31}\varepsilon_{xx}^{(0)} + \bar{e}_{32}\varepsilon_{yy}^{(0)} + \bar{e}_{36}\gamma_{xy}^{(0)} + z_c^k\left(\bar{e}_{31}\varepsilon_{xx}^{(1)} + \bar{e}_{32}\varepsilon_{yy}^{(1)} + \bar{e}_{36}\gamma_{xy}^{(1)}\right)\right]^{(k)} dxdy \tag{5.6}$$

where $z_c^k = 0.5(z_k + z_{k+1})$.

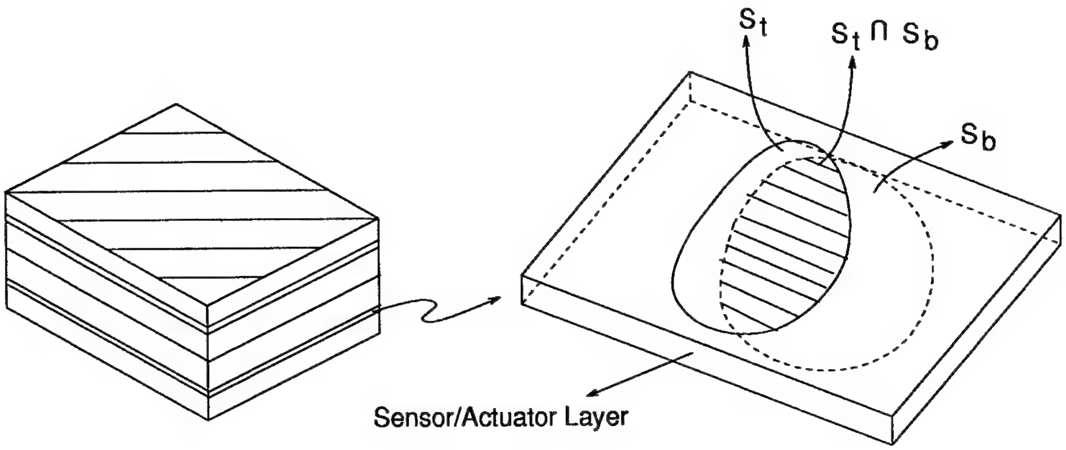

Figure 5.1: Laminate with surface electrodes (S_t =surface electrode in the top surface, S_b = surface electrode in the bottom surface.

We note that Eq. (5.6) was obtained by setting the externally applied fields $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(1)}$ to zero, because when the surface electrodes are short-circuited on both sides of the piezoelectric lamina, the terms containing the externally applied electric field intensities $\mathcal{E}_z^{(1)}$ and $\mathcal{E}_z^{(2)}$ should be set to zero. The resulting equation is the *closed-circuit charge sensor equation*, which relates the output signal to the plate deformation.

The current $I(t)$ on the surface of the sensor is given by

$$I(t) = \frac{dQ_s}{dt} \tag{5.7}$$

When the sensors are used as strain rate tensors, the current can be converted to the open circuit sensor voltage ouput V_s by[12,39]

$$V_s(t) = G_c I(t) = G_c \frac{dQ_s}{dt} \tag{5.8}$$

where G_c is the gain of the current amplifier.

The sensor forces $\{N^P\}$ and moments $\{M^P\}$ (and $\{P^P\}$) are then evaluated by using

$$\{\mathcal{E}\} = \begin{Bmatrix} 0 \\ 0 \\ \frac{V_s}{h_s} \end{Bmatrix} \tag{5.9}$$

in Eqs. (3.32) and (3.33). Here h_s denotes the thickness of the sensor layer. The forces and moments will be determined in terms of of the strain rates in the sensor layers.

5.3. Actuator equations

The actuator equations for piezoelectric actuator can be derived using induced strain actuation. We assume that the piezoelectric actuator layer does not have any applied stress. Then Eq. (5.1) gives the strains developed by the electric field on the actuator layer

$$\{\varepsilon\}_a = [\bar{Q}]^{-1}[\bar{e}]^{\mathrm{T}}\{\mathcal{E}\} = [\bar{Q}]^{-1}[\bar{Q}]^{\mathrm{T}}[\bar{d}]^{\mathrm{T}}\{\mathcal{E}\} \tag{5.10}$$

or

$$[\bar{Q}]\{\varepsilon\}_a = [\bar{Q}]^{\mathrm{T}}[\bar{d}]^{\mathrm{T}}\{\mathcal{E}\} \tag{5.11}$$

The stresses due to the actuator strains are

$$\{\sigma\}_a = [\bar{Q}]\{\varepsilon\}_a = [\bar{Q}]^{\mathrm{T}}[\bar{d}]^{\mathrm{T}}\{\mathcal{E}\} \tag{5.12a}$$

or

$$\sigma_i^a = \bar{Q}_{ij}\bar{d}_{kj}\mathcal{E}_k \qquad (i,j = 1,2,6;\; k = 1,2,3) \tag{5.12b}$$

The stress vector can be used to generate the actuator forces $\{N^P\}$ and moments $\{M^P\}$.

If the voltage applied to the actuator in the thickness direction (only) is V_a, then

$$\{\mathcal{E}\} = \begin{Bmatrix} 0 \\ 0 \\ \frac{V_a}{h_a} \end{Bmatrix} \tag{5.13}$$

where h_a is the thickness of the actuator layer. Then we have

$$N_i^P = \bar{Q}_{ij}\bar{d}_{3j}V_a \tag{5.14a}$$
$$M_i^P = \bar{Q}_{ij}\bar{d}_{3j}V_a z_c^a \tag{5.14b}$$

where z_c^a is the distance from the midplane to the center of the actuator layer.

6. The Navier solutions of CLPT

6.1. Preliminary comments

The equations of motion (3.10)–(3.12) can be expressed in terms of displacements (u_0, v_0, w_0), temperature field (T^0, T^1), and electrical field $(\mathcal{E}_z^0, \mathcal{E}_z^1)$ by substituting for the force and moment resultants from Eqs.(3.37) and (3.38). For homogeneous laminates (*i.e.*, for laminates with constant A's, B's, and D's) and under the assumption of small strains, displacements and rotations, the equations of motion (3.10)–(3.12) can be expressed as[52]

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} + \begin{bmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{Bmatrix} \ddot{u}_0 \\ \ddot{v}_0 \\ \ddot{w}_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \end{Bmatrix} + \begin{Bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{Bmatrix} + \begin{Bmatrix} f_1^P \\ f_2^P \\ f_3^P \end{Bmatrix} \tag{6.1}$$

where the coefficients c_{ij} and m_{ij} are

$$
\begin{aligned}
c_{11} &= A_{11}d_x^2 + 2A_{16}d_xd_y + A_{66}d_y^2 \\
c_{12} &= A_{16}d_x^2 + (A_{12} + A_{66})d_xd_y + A_{26}d_y^2 \\
c_{13} &= -\left[B_{11}d_x^3 + 3B_{16}d_x^2d_y + (B_{12} + 2B_{66})d_xd_y^2 + B_{26}d_y^3\right] \\
c_{22} &= A_{66}d_x^2 + 2A_{26}d_xd_y + A_{22}d_y^2 \\
c_{23} &= -\left[B_{16}d_x^3 + (B_{12} + 2B_{66})d_x^2d_y + 3B_{26}d_xd_y^2 + B_{22}d_y^3\right] \\
c_{33} &= D_{11}d_x^4 + 4D_{16}d_x^3d_y + 2(D_{12} + 2D_{66})d_x^2d_y^2 + 4D_{26}d_xd_y^3 + D_{22}d_y^4 \\
m_{11} &= -I_0d_t^2\ , \quad m_{13} = I_1d_xd_t^2\ , \quad m_{22} = -I_0d_t^2 \\
m_{23} &= I_1d_yd_t^2\ , \ m_{33} = I_0d_t^2 - I_2d_t^2(d_x^2 + d_y^2)
\end{aligned}
\tag{6.2}
$$

d_x^i, d_y^i, and d_t^i denote the differential operators

$$
d_x^i = \frac{\partial^i}{\partial x^i}\ , \quad d_y^i = \frac{\partial^i}{\partial y^i}\ , \quad d_t^i = \frac{\partial^i}{\partial t^i} \qquad (i = 1, 2, 3, 4)
\tag{6.3}
$$

and f_i^T and f_i^P are equivalent generalized thermal and piezoelectric forces

$$
\begin{aligned}
f_1^T &= \frac{\partial N_{xx}^T}{\partial x} + \frac{\partial N_{xy}^T}{\partial y}, \quad f_2^T = \frac{\partial N_{xy}^T}{\partial x} + \frac{\partial N_{yy}^T}{\partial y} \\
f_3^T &= -\left(\frac{\partial^2 M_{xx}^T}{\partial x^2} + 2\frac{\partial^2 M_{xy}^T}{\partial x \partial y} + \frac{\partial^2 M_{yy}^T}{\partial y^2}\right)
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
f_1^P &= \frac{\partial N_{xx}^P}{\partial x} + \frac{\partial N_{xy}^P}{\partial y}, \quad f_2^P = \frac{\partial N_{xy}^P}{\partial x} + \frac{\partial N_{yy}^P}{\partial y} \\
f_3^P &= -\left(\frac{\partial^2 M_{xx}^P}{\partial x^2} + 2\frac{\partial^2 M_{xy}^P}{\partial x \partial y} + \frac{\partial^2 M_{yy}^P}{\partial y^2}\right)
\end{aligned}
\tag{6.5}
$$

In this section, we develop the Navier solutions of Eq. (6.1) for simply supported rectangular laminates with actuating and/or sensing layers. In the Navier method the generalized displacements and applied loads are expanded in a double trigonometric series in terms of unknown parameters. The choice of the functions in the series is restricted to those which satisfy the boundary conditions of the problem. Substitution of the displacement expansions into the governing equations should result in a unique, invertible, set of algebraic equations among the parameters of the expansion. Otherwise, the Navier solution cannot be developed for the problem.

The Navier solutions can be developed only for two classes of laminates. The first class has the following stiffness characteristics:

$$
\begin{gathered}
A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0 \\
A_6^T = B_6^T = A_6^P = B_6^P = 0
\end{gathered}
\tag{6.6}
$$

and the second one has

$$
\begin{gathered}
A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0 \\
A_6^T = B_6^T = A_6^P = B_6^P = 0,\ I_1 = 0
\end{gathered}
\tag{6.7}
$$

These restrictions necessarily require that the actuating and sensing layers should be placed such that the laminate stiffnesses satisfy the above requirements. The stiffness characteristics in Eq. (6.6) are satisfied by antisymmetric cross-ply laminates, while those in Eq. (6.7) by antisymmetric angle-ply laminates[52].

6.2. Antisymmetric cross-ply plates

The simply supported (SS-1) boundary conditions used for antisymmetric cross-ply laminates are

$$\begin{gathered}
u_0(x,0,t)=0,\ \ \frac{\partial w_0}{\partial x}(x,0,t)=0,\ \ u_0(x,b,t)=0,\ \ \frac{\partial w_0}{\partial x}(x,b,t)=0\\
v_0(0,y,t)=0,\ \ \frac{\partial w_0}{\partial y}(0,y,t)=0,\ \ v_0(a,y,t)=0,\ \ \frac{\partial w_0}{\partial y}(a,y,t)=0\\
w_0(x,0,t)=0,\ \ w_0(x,b,t)=0,\ \ w_0(0,y,t)=0,\ \ w_0(a,y,t)=0\\
N_{xx}(0,y,t)=0,\ \ N_{xx}(a,y,t)=0,\ \ N_{yy}(x,0,t)=0,\ \ N_{yy}(x,b,t)=0\\
M_{xx}(0,y,t)=0,\ \ M_{xx}(a,y,t)=0,\ \ M_{yy}(x,0,t)=0,\ \ M_{yy}(x,b,t)=0
\end{gathered} \tag{6.8}$$

where a and b are the planform dimensions of the plate.

The displacement boundary conditions in (6.8) are satisfied by assuming the following form of the displacements:

$$u_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}U_{mn}(t)\cos\alpha x\ \sin\beta y \tag{6.9a}$$

$$v_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}V_{mn}(t)\sin\alpha x\ \cos\beta y \tag{6.9b}$$

$$w_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{mn}(t)\sin\alpha x\ \sin\beta y \tag{6.9c}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$ and (U_{mn}, V_{mn}, W_{mn}) are coefficients to be determined. Similarly, the applied loads are expanded in double sine series as

$$q(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Q_{mn}(t)\sin\alpha x\ \sin\beta y \tag{6.10a}$$

$$\{N^T\}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{N^T_{mn}\}\sin\alpha x\ \sin\beta y \tag{6.10b}$$

$$\{M^T\}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{M^T_{mn}\}\sin\alpha x\ \sin\beta y \tag{6.10c}$$

$$\{N^P\}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{N^P_{mn}\}\sin\alpha x\ \sin\beta y \tag{6.10d}$$

$$\{M^P\}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{M^P_{mn}\}\sin\alpha x\ \sin\beta y \tag{6.10e}$$

where the coefficients Q_{mn}, for example, are given by

$$Q_{mn}(t) = \frac{4}{ab} \int_0^b \int_0^a q(x,y,t)\ \sin\alpha x\ \sin\beta y\ dxdy \tag{6.11}$$

Substitution of Eqs. (6.9)-(6.11) into Eq. (6.1) yields for any m, n, x, and y, the following differential equations in time

$$\begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} + \begin{bmatrix} \hat{m}_{11} & 0 & -I_1\alpha \\ 0 & \hat{m}_{22} & -I_1\beta \\ -I_1\alpha & -I_1\beta & \hat{m}_{33} \end{bmatrix} \begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{mn} \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix} + \begin{Bmatrix} -\alpha N^T_{1mn} \\ -\beta N^T_{2mn} \\ \alpha^2 M^T_{1mn} + \beta^2 M^T_{2mn} \end{Bmatrix} + \begin{Bmatrix} -\alpha N^P_{1mn} \\ -\beta N^P_{2mn} \\ \alpha^2 M^P_{1mn} + \beta^2 M^P_{2mn} \end{Bmatrix} \tag{6.12}$$

where $\hat{c}_{ij}$ and $\hat{m}_{ij}$ are

$$\begin{aligned} &\hat{c}_{11} = (A_{11}\alpha^2 + A_{66}\beta^2), \quad \hat{c}_{12} = (A_{12} + A_{66})\alpha\beta \\ &\hat{c}_{13} = -B_{11}\alpha^3 - (B_{12} + 2B_{66})\alpha\beta^2, \quad \hat{c}_{22} = (A_{66}\alpha^2 + A_{22}\beta^2) \\ &\hat{c}_{23} = -(B_{12} + 2B_{66})\alpha^2\beta - B_{22}\beta^3 \\ &\hat{c}_{33} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 \\ &\hat{m}_{11} = I_0, \ \hat{m}_{22} = I_0, \ \hat{m}_{33} = I_0 + I_2(\alpha^2 + \beta^2) \end{aligned} \tag{6.13}$$

Equation (6.12) must be modified to include the sensor and actuator equations. If the plate has only actuators, then the modification is simple; the actuator forces $\{N^P\}$ and moments $\{M^P\}$ in the above equation must be replaced using Eqs. (6.14a,b). One must be careful to include them only for the actuator layers, and for other layers one must set $\{N^P\} = \{M^P\} = \{0\}$.

6.3. Antisymmetric angle-ply plates

The simply supported (SS-2) boundary conditions for the antisymmetric angle-ply laminates are

$$\begin{gathered} u_0(0,y,t) = 0, \quad u_0(a,y,t) = 0, \quad v_0(x,0,t) = 0, \quad v_0(x,b,t) = 0 \\ \frac{\partial w_0}{\partial x}(x,0,t) = 0, \quad \frac{\partial w_0}{\partial x}(x,b,t) = 0, \quad \frac{\partial w_0}{\partial y}(0,y,t) = 0, \quad \frac{\partial w_0}{\partial y}(a,y,t) = 0 \\ w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \\ N_{xy}(0,y,t) = 0, \quad N_{xy}(a,y,t) = 0, \quad N_{xy}(x,0,t) = 0, \quad N_{xy}(x,b,t) = 0 \\ M_{xx}(0,y,t) = 0, \quad M_{xx}(a,y,t) = 0, \quad M_{yy}(x,0,t) = 0, \quad M_{yy}(x,b,t) = 0 \end{gathered} \tag{6.14}$$

The displacement boundary conditions in (6.14) are satisfied by assuming the following form of the displacements

$$u_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn}(t)\ \sin\alpha x\ \cos\beta y \tag{6.15}$$

$$v_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn}(t)\ \cos\alpha x\ \sin\beta y \tag{6.16}$$

$$w_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t)\ \sin\alpha x\ \sin\beta y \tag{6.17}$$

Substitution of Eqs. (6.15)-(6.17) and (6.10a–d) into Eqs. (6.1) yields

$$\begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33}+\hat{s}_{33} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} + \begin{bmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{bmatrix} \begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{mn} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix} + \begin{Bmatrix} -\beta N^T_{6mn} \\ -\alpha N^T_{6mn} \\ \alpha^2 M^T_{1mn} + \beta^2 M^T_{2mn} \end{Bmatrix} + \begin{Bmatrix} -\beta N^P_{6mn} \\ -\alpha N^P_{6mn} \\ \alpha^2 M^P_{1mn} + \beta^2 M^P_{2mn} \end{Bmatrix} \tag{6.18}$$

where

$$\begin{aligned} &\hat{c}_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad \hat{c}_{12} = (A_{12}+A_{66})\alpha\beta \\ &\hat{c}_{13} = -(3B_{16}\alpha^2 + B_{26}\beta^2)\beta, \quad \hat{c}_{22} = A_{66}\alpha^2 + A_{22}\beta^2 \\ &\hat{c}_{23} = -(B_{16}\alpha^2 + 3B_{26}\beta^2)\alpha \\ &\hat{c}_{33} = D_{11}\alpha^4 + 2(D_{12}+2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 \\ &\hat{m}_{11} = I_0, \quad \hat{m}_{22} = I_0, \quad \hat{m}_{33} = I_0 + I_2(\alpha^2+\beta^2) \end{aligned} \tag{6.19}$$

7. The Navier solutions of TSDT

7.1. Preliminary comments

The five equations of motion, Eqs.(4.12)–(4.16), admit the Navier solutions for simply supported antisymmetric cross-ply and angle-ply laminates[52,56,57]. For antisymmetric cross-ply laminates the following stiffnesses are zero:

$$A_{16} = A_{26} = A_{45} = B_{16} = B_{26} = D_{16} = D_{26} = I_1 = 0$$

$$E_{16} = E_{26} = F_{16} = F_{26} = H_{16} = H_{26} = D_{45} = F_{45} = I_3 = I_5 = I_7 = 0 \tag{7.1}$$

and for antisymmetric angle-ply laminates the following stiffnesses are zero:

$$A_{16} = A_{26} = A_{45} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = I_1 = 0$$

$$B_{11} = B_{12} = B_{22} = B_{66} = F_{16} = F_{26} = H_{16} = H_{26} = D_{45} = F_{45} = I_3 = I_5 = I_7 = 0 \tag{7.2}$$

The Navier solutions of the linear equations for these two cases are presented next.

7.2. Antisymmetric cross-ply plates

The SS-1 boundary conditions for the third-order shear deformation plate theory are

$$u_0(x,0,t) = 0, \quad \phi_x(x,0,t) = 0, \quad u_0(x,b,t) = 0, \quad \phi_x(x,b,t) = 0$$

$$v_0(0,y,t) = 0, \quad \phi_y(0,y,t) = 0, \quad v_0(a,y,t) = 0, \quad \phi_y(a,y,t) = 0$$

$$w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0$$

$$N_{xx}(0,y,t) = 0, \quad N_{xx}(a,y,t) = 0, \quad N_{yy}(x,0,t) = 0, \quad N_{yy}(x,b,t) = 0$$

$$\bar{M}_{xx}(0,y,t) = 0, \quad \bar{M}_{xx}(a,y,t) = 0, \quad \bar{M}_{yy}(x,0,t) = 0, \quad \bar{M}_{yy}(x,b,t) = 0 \tag{7.3}$$

The boundary conditions in Eq. (7.3) are satisfied by the following expansions:

$$u_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn}(t)\cos\alpha x \sin\beta y \tag{7.4a}$$

$$v_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}(t)\sin\alpha x \cos\beta y \tag{7.4b}$$

$$w_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}(t)\sin\alpha x \sin\beta y \tag{7.4c}$$

$$\phi_x(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_{mn}(t)\cos\alpha x \sin\beta y \tag{7.4d}$$

$$\phi_y(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Y_{mn}(t)\sin\alpha x \cos\beta y \tag{7.4e}$$

The transverse load q, thermal forces and moments $(\{N^T\},\{M^T\},\{P^T\})$, and electrical forces and moments $(\{N^P\},\{M^P\},\{P^P\})$ are also expanded in double Fourier sine series [see Eqs.(6.10a–e)].

Substitution of Eqs.(7.4*a-e*) and applied load expansions into Eqs.(4.12)-(4.16), we obtain a 5×5 system of the following differential equations in time

$$\begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} & \hat{s}_{13} & \hat{s}_{14} & \hat{s}_{15} \\ \hat{s}_{12} & \hat{s}_{22} & \hat{s}_{23} & \hat{s}_{24} & \hat{s}_{25} \\ \hat{s}_{13} & \hat{s}_{23} & \hat{s}_{33} & \hat{s}_{34} & \hat{s}_{35} \\ \hat{s}_{14} & \hat{s}_{24} & \hat{s}_{34} & \hat{s}_{44} & \hat{s}_{45} \\ \hat{s}_{15} & \hat{s}_{25} & \hat{s}_{35} & \hat{s}_{45} & \hat{s}_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} + \begin{bmatrix} \hat{m}_{11} & 0 & 0 & 0 & 0 \\ 0 & \hat{m}_{22} & 0 & 0 & 0 \\ 0 & 0 & \hat{m}_{33} & \hat{m}_{34} & \hat{m}_{35} \\ 0 & 0 & \hat{m}_{43} & \hat{m}_{44} & 0 \\ 0 & 0 & \hat{m}_{53} & 0 & \hat{m}_{55} \end{bmatrix} \begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{mn} \\ \ddot{X}_{mn} \\ \ddot{Y}_{mn} \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \alpha N^T_{1mn} \\ \beta N^T_{2mn} \\ 0 \\ \alpha M^T_{1mn} \\ \beta M^T_{2mn} \end{Bmatrix} - \begin{Bmatrix} \alpha N^P_{1mn} \\ \beta N^P_{2mn} \\ 0 \\ \alpha M^P_{1mn} \\ \beta M^P_{2mn} \end{Bmatrix} \tag{7.5}$$

where $\hat{s}_{ij}$ and $\hat{m}_{ij}$ are defined by

$$\hat{s}_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad \hat{s}_{12} = (A_{12}+A_{66})\alpha\beta$$
$$\hat{s}_{13} = -c_1\left[E_{11}\alpha^2 + (E_{12}+2E_{66})\beta^2\right]\alpha$$
$$\hat{s}_{14} = \hat{B}_{11}\alpha^2 + \hat{B}_{66}\beta^2, \quad \hat{s}_{15} = (\hat{B}_{12}+\hat{B}_{66})\alpha\beta$$
$$\hat{s}_{22} = A_{66}\alpha^2 + A_{22}\beta^2, \quad \hat{s}_{24} = \hat{s}_{15}$$
$$\hat{s}_{23} = -c_1\left[E_{22}\beta^2 + (E_{12}+2E_{66})\alpha^2\right]\beta, \quad \hat{s}_{25} = \hat{B}_{66}\alpha^2 + \hat{B}_{22}\beta^2$$
$$\hat{s}_{33} = \bar{A}_{55}\alpha^2 + \bar{A}_{44}\beta^2 + c_1^2\left[H_{11}\alpha^4 + 2(H_{12}+2H_{66})\alpha^2\beta^2 + H_{22}\beta^4\right]$$
$$\hat{s}_{34} = \bar{A}_{55}\alpha - c_1\left[\hat{F}_{11}\alpha^3 + (\hat{F}_{12}+2\hat{F}_{66})\alpha\beta^2\right]$$
$$\hat{s}_{35} = \bar{A}_{44}\beta - c_1\left[\hat{F}_{22}\beta^3 + (\hat{F}_{12}+2\hat{F}_{66})\alpha^2\beta\right]$$
$$\hat{s}_{44} = \bar{A}_{55} + \bar{D}_{11}\alpha^2 + \bar{D}_{66}\beta^2, \quad \hat{s}_{45} = (\bar{D}_{12}+\bar{D}_{66})\alpha\beta$$

$$\hat{s}_{55} = \bar{A}_{44} + \bar{D}_{66}\alpha^2 + \bar{D}_{22}\beta^2 \tag{7.6a}$$

$$\hat{m}_{11} = I_0, \quad \hat{m}_{22} = I_0, \quad \hat{m}_{33} = I_0 + c_1^2 I_6\left(\alpha^2 + \beta^2\right), \quad \hat{m}_{34} = -c_1 J_4 \alpha$$

$$\hat{m}_{35} = -c_1 J_4 \beta, \quad \hat{m}_{44} = K_2, \quad \hat{m}_{55} = K_2 \tag{7.6b}$$

$$\hat{A}_{ij} = A_{ij} - c_1 D_{ij}, \quad \hat{B}_{ij} = B_{ij} - c_1 E_{ij}, \quad \hat{D}_{ij} = D_{ij} - c_1 F_{ij} \quad (i,j = 1,2,6)$$

$$\hat{F}_{ij} = F_{ij} - c_1 H_{ij}, \quad \bar{A}_{ij} = \hat{A}_{ij} - c_1 \hat{D}_{ij} = A_{ij} - 2c_1 D_{ij} + c_1^2 F_{ij} \quad (i,j = 1,2,6)$$

$$\bar{D}_{ij} = \hat{D}_{ij} - c_1 \hat{F}_{ij} = D_{ij} - 2c_1 F_{ij} + c_1^2 H_{ij} \quad (i,j = 1,2,6)$$

$$\bar{A}_{ij} = \hat{A}_{ij} - 3c_1 \hat{D}_{ij} = A_{ij} - 6c_1 D_{ij} + 9c_1^2 F_{ij} \quad (i,j = 4,5) \tag{7.6c}$$

The ordinary differential equations (7.5) in time can be solved for transient response using the Newmark integration procedure.

7.3. Antisymmetric angle-ply plates

The SS-2 boundary conditions for the third-order shear deformation plate theory are

$$\begin{gathered}
u_0(0,y,t) = 0, \quad u_0(a,y,t) = 0, \quad v_0(x,0,t) = 0, \quad v_0(x,b,t) = 0 \\
\phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(0,y,t) = 0, \quad \phi_y(a,y,t) = 0 \\
w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \\
N_{xy}(0,y,t) = 0, \quad N_{xy}(a,y,t) = 0, \quad N_{xy}(x,0,t) = 0, \quad N_{xy}(x,b,t) = 0 \\
\bar{M}_{xx}(0,y,t) = 0, \quad \bar{M}_{xx}(a,y,t) = 0, \quad \bar{M}_{yy}(x,0,t) = 0, \quad \bar{M}_{yy}(x,b,t) = 0
\end{gathered} \tag{7.7}$$

The simply supported (SS-2) boundary conditions in Eq.(7.7) are satisfied by the displacement expansions

$$u_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn}(t)\cos\alpha x \sin\beta y \tag{7.8a}$$

$$v_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}(t)\sin\alpha x \cos\beta y \tag{7.8b}$$

and the expansions of (w_0, ϕ_x, ϕ_y) in Eqs.(7.4c,d,e). Substituting these expansions along with the load expansions into Eqs. (4.12)-(4.16), we obtain equations of the form in (7.5)

$$[\hat{S}]\{\Delta\} + [M]\{\ddot{\Delta}\} = \{F\} \tag{7.9}$$

with the following definition of the coefficients:

$$\hat{s}_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad \hat{s}_{12} = (A_{12} + A_{66})\alpha\beta, \quad \hat{s}_{13} = -c_1\left(3E_{16}\alpha^2 + E_{26}\beta^2\right)\beta$$

$$\hat{s}_{14} = 2\hat{B}_{16}\alpha\beta, \quad \hat{s}_{15} = \hat{B}_{16}\alpha^2 + \hat{B}_{26}\beta^2, \quad \hat{s}_{22} = A_{66}\alpha^2 + A_{22}\beta^2$$

$$\hat{s}_{23} = -c_1\left(E_{16}\alpha^2 + 3E_{26}\beta^2\right)\alpha, \quad \hat{s}_{24} = \hat{s}_{15}, \quad \hat{s}_{25} = 2\hat{B}_{25}\alpha\beta$$

$$\hat{s}_{33} = \bar{A}_{55}\alpha^2 + \bar{A}_{44}\beta^2 + c_1^2\left[H_{11}\alpha^4 + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + H_{22}\beta^4\right]$$

$$\hat{s}_{34} = \bar{A}_{55}\alpha - c_1\left[\hat{F}_{11}\alpha^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha\beta^2\right]$$

$$\hat{s}_{35} = \bar{A}_{44}\alpha - c_1\left[\hat{F}_{22}\beta^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha^2\beta\right]$$

$$\hat{s}_{44} = \bar{A}_{55} + \bar{D}_{11}\alpha^2 + \bar{D}_{66}\beta^2, \quad \hat{s}_{45} = (\bar{D}_{12} + \bar{D}_{66})\alpha\beta, \quad \hat{s}_{55} = \bar{A}_{44} + \bar{D}_{66}\alpha^2 + \bar{D}_{22}\beta^2 \tag{7.10}$$

The mass coefficients are the same as those defined in Eq. (7.6b), except for $\hat{m}_{34} = \hat{m}_{35} = 0$. The vector $\{F\}$ is given by

$$\{F\} = \left\{ \begin{array}{c} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{array} \right\} - \left\{ \begin{array}{c} \alpha N^T_{6mn} \\ \beta N^T_{6mn} \\ 0 \\ \alpha M^T_{1mn} \\ \beta M^T_{2mn} \end{array} \right\} - \left\{ \begin{array}{c} \alpha N^P_{6mn} \\ \beta N^P_{6mn} \\ 0 \\ \alpha M^P_{1mn} \\ \beta M^P_{2mn} \end{array} \right\} \tag{7.11}$$

8. Finite element formulation of CLPT

8.1. Weak forms

In this section, finite element models of Eqs.(3.10)–(3.12) governing the motion of laminated plates with piezoelectric layers are developed for the linear case first. Then the nonlinear finite element model will be discussed. The finite element model is based on the weak forms of Eqs.(3.10)–(3.12). The weak forms are (see Reddy[52], Chapter 10)

$$\begin{aligned} 0 = &\int_{\Omega^e} \left[\frac{\partial \delta u_0}{\partial x} N_{xx} + \frac{\partial \delta u_0}{\partial y} N_{xy} + I_0 \delta u_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \delta u_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right)\right] dxdy \\ &- \oint_{\Gamma^e} \left(N_{xx} n_x + N_{xy} n_y\right) \delta u_0 \; ds \end{aligned} \tag{8.1a}$$

$$\begin{aligned} 0 = &\int_{\Omega^e} \left[\frac{\partial \delta v_0}{\partial x} N_{xy} + \frac{\partial \delta v_0}{\partial y} N_{yy} + I_0 \delta v_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \delta v_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right)\right] dxdy \\ &- \oint_{\Gamma^e} \left(N_{xy} n_x + N_{yy} n_y\right) \delta v_0 \; ds \end{aligned} \tag{8.1b}$$

$$\begin{aligned} 0 = &\int_{\Omega^e} \left[-\frac{\partial^2 \delta w_0}{\partial x^2} M_{xx} - 2\frac{\partial^2 \delta w_0}{\partial x \partial y} M_{xy} - \frac{\partial^2 \delta w_0}{\partial y^2} M_{yy} - \delta w_0 q \right. \\ &+ I_0 \delta w_0 \frac{\partial^2 w_0}{\partial t^2} + I_2 \left(\frac{\partial \delta w_0}{\partial x}\frac{\partial^3 w_0}{\partial x \partial t^2} + \frac{\partial \delta w_0}{\partial y}\frac{\partial^3 w_0}{\partial y \partial t^2}\right) \\ &\left. - I_1 \left(\frac{\partial \delta w_0}{\partial x}\frac{\partial^2 u_0}{\partial t^2} + \frac{\partial \delta w_0}{\partial y}\frac{\partial^2 v_0}{\partial t^2}\right)\right] dxdy - \oint_{\Gamma^e} V_n \delta w_0 \; ds \\ &+ \oint_{\Gamma^e} \left[\frac{\partial \delta w_0}{\partial x}\left(M_{xx} n_x + M_{xy} n_y\right) + \frac{\partial \delta w_0}{\partial y}\left(M_{xy} n_x + M_{yy} n_y\right)\right] ds \end{aligned} \tag{8.1c}$$

where (n_x, n_y) denote the direction cosines of the unit normal on the element boundary Γ^e of the element, $(\delta u_0, \delta v_0, \delta w_0)$ are the virtual displacements, which take the role of weight functions, and V_n is the effective shear force defined in Eq. (2.16). The stress and moment resultants N_{xx}, M_{xx}, etc. are known in terms of the displacements (u_0, v_0, w_0) and thermal and piezoelectric forces and moments through Eqs. (3.30)–(3.33).

8.2. Spatial approximations

First, we note that the stress and moment resultants contain first-order derivatives of (u_0, v_0) and second-order derivatives of w_0 with respect to the coordinates x and y.

Second, the primary variables u_0, v_0, w_0, $\partial w_0/\partial x$, and $\partial w_0/\partial y$ must be carried as the nodal variables in order to enforce their interelement continuity. Thus, the displacements (u_0, v_0) must be approximated using the Lagrange interpolation functions, whereas w_0 should be approximated using Hermite interpolation functions over an element Ω^e. Let

$$u_0(x,y,t) \approx \sum_{j=1}^{m} u_j^e(t)\psi_j^e(x,y) \tag{8.2a}$$

$$v_0(x,y,t) \approx \sum_{j=1}^{m} v_j^e(t)\psi_j^e(x,y) \tag{8.2b}$$

$$w_0(x,y,t) \approx \sum_{k=1}^{n} \Delta_k^e(t)\varphi_k^e(x,y) \tag{8.2c}$$

where (u_j^e, v_j^e) denote the values of (u_0, v_0) at the jth node of the Lagrange elements, Δ_k^e denote the values of w_0 and its derivatives with respect to x and y at the kth node, and (ψ_j^e, φ_k^e) are the Lagrange and Hermite interpolation functions, respectively.

There exists vast literature on triangular and rectangular plate bending finite elements of isotropic or orthotropic plates based on the classical plate theory[52]. Here we discuss rectangular C^1 plate bending elements. There are two kinds of C^1 plate bending elements. A *conforming element* is one in which the interelement continuity of w_0, $\theta_x \equiv \partial w_0/\partial x$, and $\theta_y \equiv \partial w_0/\partial y$ (or $\partial w_0/\partial n$) are satisfied, and a *nonconforming element* is one in which the continuity of the normal slope, $\partial w_0/\partial n$, is not satisfied (see Figure 8.1).

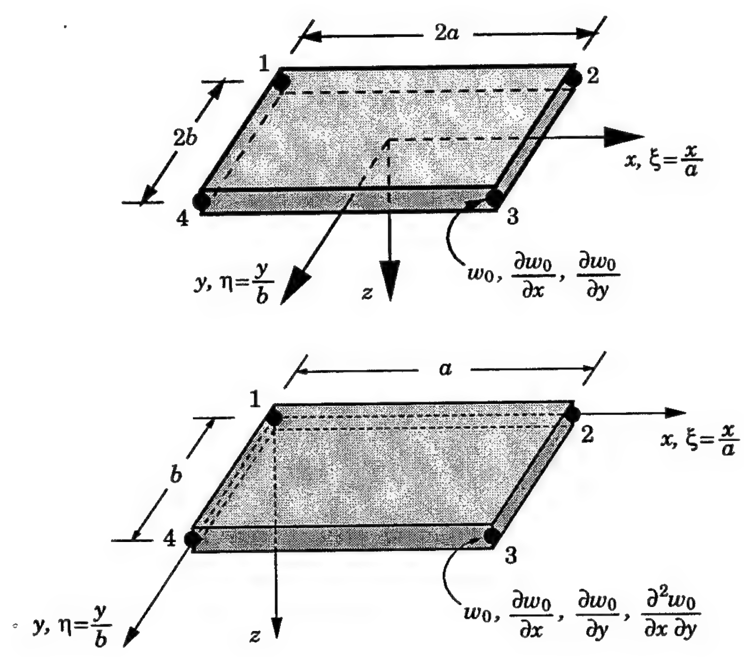

Figure 8.1: (a) Nonconforming and (b) conforming finite elements for plate bending based the classical plate theory.

A nonconforming rectangular element has w_0, θ_x, and θ_y as the nodal variables (see Figure 8.1a). The normal slope varies cubically along an edge whereas there are only two values of $\partial w_0/\partial n$ available on the edge. Therefore, the cubic polynomial for the normal derivative of w_0 is not the same on the edge common to two elements. The interpolation functions for this element can be expressed compactly as

$$\varphi_i^e = g_{i1}\ (i = 1,4,7,10);\quad \varphi_i^e = g_{i2}\ (i = 2,5,8,11);\quad \varphi_i^e = g_{i3}\ (i = 3,6,9,12) \tag{8.3a}$$

where

$$\begin{aligned} g_{i1} &= \frac{1}{8}(1+\xi_0)(1+\eta_0)(2+\xi_0+\eta_0-\xi^2-\eta^2) \\ g_{i2} &= \frac{1}{8}\xi_i(\xi_0-1)(1+\eta_0)(1+\xi_0)^2,\quad g_{i3} = \frac{1}{8}\eta_i(\eta_0-1)(1+\xi_0)(1+\eta_0)^2 \\ \xi &= (x-x_c)/2,\quad \eta = (y-y_c)/b,\quad \xi_0 = \xi\xi_i,\ \eta_0 = \eta\eta_i \end{aligned} \tag{8.3b}$$

where $2a$ and $2b$ are the sides of the rectangle, and (x_c, y_c) are the global coordinates of the center of the rectangle.

A conforming rectangular element with w_0, $\partial w_0/\partial x$, $\partial w_0/\partial y$, and $\partial^2 w_0/\partial x\partial y$ as the nodal variables is shown in Figure 8.1b. The interpolation functions for this element are

$$\begin{aligned} &\varphi_i^e = g_{i1}\ (i = 1,5,9,13);\quad \varphi_i^e = g_{i2}\ (i = 2,6,10,14) \\ &\varphi_i^e = g_{i3}\ (i = 3,7,11,15);\quad \varphi_i^e = g_{i4}\ (i = 4,8,12,16) \end{aligned} \tag{8.4a}$$

where

$$\begin{aligned} g_{i1} &= \frac{1}{16}(\xi+\xi_i)^2(\xi_0-2)(\eta+\eta_i)^2(\eta_0-2),\quad g_{i2} = \frac{1}{16}\xi_i(\xi+\xi_i)^2(1-\xi_0)(\eta+\eta_i)^2(\eta_0-2) \\ g_{i3} &= \frac{1}{16}\eta_i(\xi+\xi_i)^2(\xi_0-2)(\eta+\eta_i)^2(1-\eta_0),\quad g_{i4} = \frac{1}{16}\xi_i\eta_i(\xi+\xi_i)^2(1-\xi_0)(\eta+\eta_i)^2(1-\eta_0) \end{aligned} \tag{8.4b}$$

The conforming element has a total of six degrees of freedom per node, whereas the nonconforming element has a total of five degrees of freedom per node. For the conforming rectangular element ($m = 4$ and $n = 12$) the total number of nodal degrees of freedom per element is 24, and the nonconforming element the total number of degrees of freedom per element is 12.

8.3. Semidiscrete finite element model

Substituting approximations (8.2) for the displacements and the ith interpolation function for the virtual displacement ($\delta u_0 \sim \psi_i, \delta v_0 \sim \psi_i, \delta w_0 \sim \varphi_i$) into the weak forms, we obtain the ith equation associated with each weak form

$$\begin{aligned} 0 &= \sum_{j=1}^{m}\left(K_{ij}^{11}u_j^e + K_{ij}^{12}v_j^e + M_{ij}^{11}\ddot{u}_j^e\right) + \sum_{k=1}^{n}\left(K_{ik}^{13}\Delta_k^e + M_{ik}^{13}\ddot{\Delta}_k^e\right) - F_i^1 - F_i^{T1} - F_k^{P1} \\ 0 &= \sum_{j=1}^{m}\left(K_{ij}^{21}u_j^e + K_{ij}^{22}v_j^e + M_{ij}^{22}\ddot{v}_j^e\right) + \sum_{k=1}^{n}\left(K_{ik}^{23}\Delta_k^e + M_{ik}^{23}\ddot{\Delta}_k^e\right) - F_i^2 - F_i^{T2} - F_k^{P2} \\ 0 &= \sum_{j=1}^{m}\left(K_{kj}^{31}u_j^e + K_{kj}^{32}v_j^e + M_{kj}^{31}\ddot{u}_j^e + M_{kj}^{32}\ddot{v}_j^e\right) + \sum_{\ell=1}^{n}\left(K_{k\ell}^{33}\Delta_\ell^e + M_{k\ell}^{33}\ddot{\Delta}_\ell^e\right) \\ &\quad - F_k^3 - F_k^{T3} - F_k^{P3} \end{aligned} \tag{8.5}$$

where $i = 1, 2, \cdots, m; k = 1, 2, \cdots, n$. The coefficients of the stiffness matrix $K_{ij}^{\alpha\beta} = K_{ji}^{\beta\alpha}$, mass matrix $M_{ij}^{\alpha\beta} = M_{ji}^{\beta\alpha}$ (symmetric), and force vectors F_i^{α}, $F_i^{T\alpha}$, and $F_i^{P\alpha}$ are defined as follows:

$$
\begin{aligned}
K_{ij}^{11} &= A_{11}S_{ij}^{xx} + A_{16}\left(S_{ij}^{xy} + S_{ij}^{yx}\right) + A_{66}S_{ij}^{yy}\\
K_{ij}^{12} &= A_{12}S_{ij}^{xy} + A_{16}S_{ij}^{xx} + A_{26}S_{ij}^{yy} + A_{66}S_{ij}^{yx}\\
K_{ij}^{22} &= A_{66}S_{ij}^{xx} + A_{26}\left(S_{ij}^{xy} + S_{ij}^{yx}\right) + A_{22}S_{ij}^{yy}\\
K_{ik}^{13} &= -B_{11}R_{ik}^{xxx} - B_{12}R_{ik}^{xyy} - 2B_{26}R_{ik}^{xxy} - B_{16}R_{ik}^{yxx} - B_{26}R_{ik}^{yyy} - 2B_{66}R_{ik}^{yxy}\\
K_{ik}^{23} &= -B_{16}R_{ik}^{xxx} - B_{26}R_{ik}^{xyy} - 2B_{66}R_{ik}^{xxy} - B_{12}R_{ik}^{yxx} - B_{22}R_{ik}^{yyy} - 2B_{26}R_{ik}^{yxy}\\
K_{k\ell}^{33} &= D_{11}T_{k\ell}^{xxxx} + D_{12}\left(T_{k\ell}^{xxyy} + T_{k\ell}^{yyxx}\right) + 2D_{16}\left(T_{k\ell}^{xxxy} + T_{k\ell}^{xyxx}\right)\\
&\quad + 2D_{26}\left(T_{k\ell}^{xyyy} + T_{k\ell}^{yyxy}\right) + 4D_{66}T_{k\ell}^{xyxy} + D_{22}T_{k\ell}^{yyyy}\\
M_{ij}^{11} &= \int_{\Omega^e} I_0\psi_i^e\psi_j^e \, dxdy, \quad M_{ik}^{13} = -\int_{\Omega^e} I_1\psi_i^e\frac{\partial\varphi_k^e}{\partial x}\, dxdy\\
M_{ij}^{22} &= \int_{\Omega^e} I_0\psi_i^e\psi_j^e \, dxdy, \quad M_{ik}^{23} = -\int_{\Omega^e} I_1\psi_i^e\frac{\partial\varphi_k^e}{\partial y}\, dxdy\\
M_{k\ell}^{33} &= \int_{\Omega^e}\left[I_0\varphi_k^e\varphi_\ell^e + I_2\left(\frac{\partial\varphi_k^e}{\partial x}\frac{\partial\varphi_\ell^e}{\partial x} + \frac{\partial\varphi_k^e}{\partial y}\frac{\partial\varphi_\ell^e}{\partial y}\right)\right] dxdy\\
F_i^1 &= \oint_{\Gamma^e}\left(N_{xx}n_x + N_{xy}n_y\right)\psi_i^e \, ds, \quad F_i^2 = \int_{\Gamma_e}\left(N_{xy}n_x + N_{yy}n_y\right)\psi_i^e \, ds\\
F_k^3 &= \int_{\Omega^e} q\varphi_k^e \, dxdy + \oint_{\Gamma^e}\left(V_n\varphi_k^e + T_x\frac{\partial\varphi_k^e}{\partial x} + T_y\frac{\partial\varphi_k^e}{\partial y}\right) ds\\
F_i^{T1} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xx}^T + \frac{\partial\psi_i^e}{\partial y}N_{xy}^T\right) dxdy, \quad F_i^{T2} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xy}^T + \frac{\partial\psi_i^e}{\partial y}N_{yy}^T\right) dxdy\\
F_k^{T3} &= \int_{\Omega^e}\left(\frac{\partial^2\varphi_k^e}{\partial x^2}M_{xx}^T + 2\frac{\partial^2\varphi_k^e}{\partial x\partial y}M_{xy}^T + \frac{\partial^2\varphi_k^e}{\partial y^2}M_{yy}^T\right) dxdy\\
F_i^{P1} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xx}^P + \frac{\partial\psi_i^e}{\partial y}N_{xy}^P\right) dxdy, \quad F_i^{P2} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xy}^P + \frac{\partial\psi_i^e}{\partial y}N_{yy}^P\right) dxdy\\
F_k^{P3} &= \int_{\Omega^e}\left(\frac{\partial^2\varphi_k^e}{\partial x^2}M_{xx}^P + 2\frac{\partial^2\varphi_k^e}{\partial x\partial y}M_{xy}^P + \frac{\partial^2\varphi_k^e}{\partial y^2}M_{yy}^P\right) dxdy
\end{aligned}
\tag{8.6a}
$$

and

$$
\begin{aligned}
S_{ij}^{\xi\eta} &= \int_{\Omega^e}\frac{\partial\psi_i^e}{\partial\xi}\frac{\partial\psi_j^e}{\partial\eta}\, dxdy\,, \quad R_{ik}^{\xi\eta\zeta} = \int_{\Omega^e}\frac{\partial\psi_i^e}{\partial\xi}\frac{\partial^2\varphi_k^e}{\partial\eta\partial\zeta}\, dxdy\\
T_{k\ell}^{\xi\eta\zeta\mu} &= \int_{\Omega^e}\frac{\partial^2\varphi_k^e}{\partial\xi\partial\eta}\frac{\partial^2\varphi_\ell^e}{\partial\zeta\partial\mu}\, dxdy
\end{aligned}
\tag{8.6b}
$$

and ξ, η, ζ, and μ can be equal to x or y. In matrix notation, Eq. (8.5) can be expressed as

$$
\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] & [K^{23}] \\ [K^{13}]^{\mathrm{T}} & [K^{23}]^{\mathrm{T}} & [K^{33}] \end{bmatrix}
\begin{Bmatrix} \{u^e\} \\ \{v^e\} \\ \{\Delta^e\} \end{Bmatrix}
+
\begin{bmatrix} [M^{11}] & [0] & [M^{13}] \\ [0] & [M^{22}] & [M^{23}] \\ [M^{13}]^{\mathrm{T}} & [M^{23}]^{\mathrm{T}} & [M^{33}] \end{bmatrix}
\begin{Bmatrix} \{\ddot{u}^e\} \\ \{\ddot{v}^e\} \\ \{\ddot{\Delta}^e\} \end{Bmatrix}
= \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} + \begin{Bmatrix} \{F^{T1}\} \\ \{F^{T2}\} \\ \{F^{T3}\} \end{Bmatrix} + \begin{Bmatrix} \{F^{P1}\} \\ \{F^{P2}\} \\ \{F^{P3}\} \end{Bmatrix}
\tag{8.7}
$$

8.5. Nonlinear finite element model

The nonlinear finite element model has the form

$$[K^e]\{\bar{\Delta}^e\} + [M^e]\{\ddot{\bar{\Delta}}^e\} = \{F^e\} \tag{8.8a}$$

$$[K^e] = [K_L^e + K_{NL}^e + K_{NT}^e + K_{NP}^e] \tag{8.8b}$$

where $[K_L^e]$ is the linear stiffness matrix, and $[K_{NL}^e]$ is the geometric nonlinear part, $[K_{NT}^e]$ the thermal nonlinear part, and $[K_{NP}^e]$ the piezoelectric nonlinear part of the stiffness matrix due to the nonlinear terms in the equations of motion. The elements of the linear stiffness matrix $[K_L^e]$ and mass matrix $[M^e]$, which is linear, were given in Eqs.(8.6a,b). The nonzero nonlinear stiffness coefficients $[K^{\alpha 3}]$ $(\alpha = 1,2,3)$ are

$$\left(K_{NL}^{13}\right)_{ij} = \frac{1}{2}\int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}\bar{N}_{1j} + \frac{\partial\psi_i}{\partial y}\bar{N}_{6j}\right)\,dxdy$$
$$\left(K_{NL}^{23}\right)_{ij} = \frac{1}{2}\int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}\bar{N}_{6j} + \frac{\partial\psi_i}{\partial y}\bar{N}_{2j}\right)\,dxdy$$
$$\left(K_{NL}^{31}\right)_{ij} = \int_{\Omega^e}\left(\bar{N}_{1i}\frac{\partial\psi_j}{\partial x} + \bar{N}_{6i}\frac{\partial\psi_j}{\partial y}\right)\,dxdy = 2\bar{K}_{ji}^{13}$$
$$\left(K_{NL}^{32}\right)_{ij} = \int_{\Omega^e}\left(\bar{N}_{6i}\frac{\partial\psi_j}{\partial x} + \bar{N}_{2i}\frac{\partial\psi_j}{\partial y}\right)\,dxdy = 2\bar{K}_{ji}^{23}$$
$$\left(K_{NL}^{33}\right)_{ij} = \int_{\Omega^e}\left[\frac{\partial\varphi_i}{\partial x}\left(\bar{N}_1\frac{\partial\varphi_j}{\partial x} + \bar{N}_6\frac{\partial\varphi_j}{\partial y}\right) + \frac{\partial\varphi_i}{\partial y}\left(\bar{N}_6\frac{\partial\varphi_j}{\partial x} + \bar{N}_2\frac{\partial\varphi_j}{\partial y}\right)\right]\,dxdy \tag{8.9a}$$
$$\bar{N}_{\alpha j} = A_{\alpha 1}\frac{\partial w_0}{\partial x}\frac{\partial\varphi_j}{\partial x} + A_{\alpha 2}\frac{\partial w_0}{\partial y}\frac{\partial\varphi_j}{\partial y} + A_{\alpha 6}\left(\frac{\partial w_0}{\partial x}\frac{\partial\varphi_j}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial\varphi_j}{\partial x}\right)$$
$$\bar{N}_\alpha = \frac{1}{2}\left[A_{1\alpha}\left(\frac{\partial w_0}{\partial x}\right)^2 + A_{2\alpha}\left(\frac{\partial w_0}{\partial y}\right)^2 + 2A_{6\alpha}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right] \tag{8.9b}$$
$$\left(K_{NT}^{33}\right)_{ij} = \int_{\Omega^e}\left[N_{xx}^T\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + N_{yy}^T\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y} + N_{xy}^T\left(\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial y} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial x}\right)\right]dxdy \tag{8.10}$$
$$\left(K_{NP}^{33}\right)_{ij} = \int_{\Omega^e}\left[N_{xx}^P\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + N_{yy}^P\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y} + N_{xy}^P\left(\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial y} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial x}\right)\right]dxdy \tag{8.11}$$

for $\alpha = 1,2,6$. The same expressions hold for the first–order plate theory with φ_i replaced by ψ_i, because w_0 is also approximated by the Lagrange functions ψ_i in FSDT. We note that the nonlinearity of the coefficients $[K^{13}]$, $[K^{23}]$, $[K^{31}]$, $[K^{32}]$, and $[K^{33}]$ is solely due to the transverse deflection, w_0, as can be seen from Eqs. $(8.9a,b)$. In addition, the stiffness matrix $[K^e]$ is not symmetric for the nonlinear case because

$$K_{ij}^{3\alpha} = 2K_{ji}^{\alpha 3}, \qquad \alpha = 1,2 \tag{8.12}$$

The nonlinear algebraic equations of the bending problem or fully discretized transient problem must be solved by an iterative method. In iterative methods, the nonlinear equations are linearized by evaluating the nonlinear terms with the known solution from preceding iteration(s). Two commonly used iterative methods are: (1) the Picard method, and (2) the Newton–Raphson method. While the Picard method is simple both in concept and computer implementation, the method uses the stiffness matrix $[\hat{K}^e]$, which is unsymmetric. The

Newton–Raphson method is based on the Taylor series expansion, and it uses the tangent stiffness matrix, which is symmetric for all structural problems. Also, the method has faster convergence for most applications than the Picard method. Here a brief discussion of these two iterative methods is presented. For the sake of discussion, we use the fully discretized equation

$$[\hat{K}^e(\{\Delta^e\})]\{\Delta^e\} = \{\hat{F}^e\} \tag{8.13}$$

is used.

In the Picard method, also known as the direct iteration method, the solution vector from the previous iteration is used to evaluate the stiffness matrix, and the solution at the subsequent iteration is determined by solving the assembled equations after the imposition of boundary and initial conditions. At the element level, the Picard iteration scheme may be expressed as

$$[\hat{K}^e(\{\Delta^e\}^r)]\{\Delta^e\}^{r+1} = \{\hat{F}^e\} \tag{8.14}$$

where $\{\Delta^e\}^r$ denotes the solution vector at the rth iteration. Thus, in the direct iteration method, the coefficients $\hat{K}^e_{ij}$ are obtained by evaluating them with $w_0(x,y,t)$ from the rth iteration. At the beginning of the iteration (*i.e.*, $r=0$), we assume that $\{\Delta^e\}^0 = \{0\}$ so that the solution at the first iteration is the linear solution, because the nonlinear stiffness matrix reduces to the linear one. The iteration process is continued until the difference between $\{\Delta^e\}^r$ and $\{\Delta^e\}^{r+1}$ reduces to a preselected error tolerance. The global *error criterion* is of the form

$$\sqrt{\frac{\sum_{I=1}^{N}|\Delta_I^{r+1} - \Delta_I^r|^2}{\sum_{I=1}^{N}|\Delta_I^{r+1}|^2}} < \epsilon \quad (\text{say } 10^{-3}) \tag{8.15}$$

where N is the total number of nodal generalized displacements in the finite element mesh, and ϵ is the error tolerance. All quantities in Eq. (8.15) are understood to be at the global level.

The Newton–Raphson iterative method is based on Taylor's series expansion of the nonlinear algebraic equation (8.13) about the known solution. Suppose that the solution at the rth iteration, $\{\Delta^e\}^r$, is known. Let

$$\{R^e\} \equiv [\hat{K}^e]\{\Delta^e\} - \{\hat{F}^e\} = 0 \tag{8.16}$$

where $\{R^e\}$ is called the *residual*, which is a nonlinear function of the unknown solution $\{\Delta^e\}$. Expanding $\{R^e\}$ in Taylor's series about $\{\Delta^e\}^r$, we obtain

$$\begin{aligned}\{0\} = \{R^e\} = \{R^e\}^r &+ \left[\frac{\partial\{R^e\}}{\partial\{\Delta^e\}}\right]^r (\{\Delta^e\}^{r+1} - \{\Delta^e\}^r) \\ &+ \frac{1}{2!}\left[\frac{\partial^2\{R^e\}}{\partial\{\Delta^e\}\partial\{\Delta^e\}}\right]^r (\{\Delta^e\}^{r+1} - \{\Delta^e\}^r)^2 + \ldots\end{aligned} \tag{8.17a}$$

or

$$0 = \{R^e\}^r + \left([\hat{K}^e]_r\right)^{tan}\{\delta\Delta^e\} + O(\{\delta\Delta^e\}^2) \tag{8.17b}$$

where $O(\cdot)$ denotes the higher-order terms in $\{\delta\Delta^e\}$, and $[\hat{K}^e]^{tan}$ is the *tangent stiffness matrix* (or geometric stiffness matrix)

$$\left([\hat{K}^e]_r\right)^{tan} \equiv \left[\frac{\partial\{R^e\}}{\partial\{\Delta^e\}}\right] \text{ evaluated at } \{\Delta^e\} = \{\Delta^e\}^r \tag{8.18}$$

and the increment of the solution vector is defined by

$$\{\delta\Delta^e\} = \{\Delta^e\}^{r+1} - \{\Delta^e\}^r \tag{8.19}$$

Equation (8.17b) is approximated by neglecting terms of order 2 and higher in the solution increment $\{\delta\Delta^e\}$. We obtain

$$\left([\hat{K}^e(\{\Delta^e\}^r)]\right)^{tan}\{\delta\Delta^e\} = -\{R^e\}^r \tag{8.20}$$

The assembled equations are then solved after imposing the boundary and initial conditions of the problem. We have

$$\begin{aligned}\{\delta\Delta\} &= -\left([\hat{K}(\{\Delta\}^r)]^{tan}\right)^{-1}\{R\}^r \\ &= \left([\hat{K}(\{\Delta\}^r)]^{tan}\right)^{-1}\left(\{\hat{F}\} - [\hat{K}(\{\Delta\}^r)]\{\Delta\}^r\right)\end{aligned} \tag{8.21}$$

and the total solution at the $(r+1)$th iteration is given by

$$\{\Delta\}^{r+1} = \{\Delta\}^r + \{\delta\Delta\} \tag{8.22}$$

The iteration process is continued by solving Eq. (8.20) until the convergence criteria in Eq. (8.15) is satisfied or the residual $\{R\}$, measured in the same way as the solution error in Eq. (8.15), satisfies the error criterion. Note that at the beginning of each iteration the tangent stiffness matrix and residual vector must be updated using the latest available solution $\{\Delta\}$. If the tangent stiffness matrix is kept constant for a preselected number of iterations but the residual vector is updated during the iteration, there can be a computational saving. Such an approach is called the *modified Newton–Raphson method.*

9. Finite element model of TSDT

9.1. Introduction

The primary variables of the third-order theory are $(u_n, u_s, w_0, w_{0,n} = \partial w_0/\partial n, \phi_n, \phi_s)$, where (u_n, u_s) denote in-plane normal and tangential displacements, and (ϕ_n, ϕ_s) are the rotations of a transverse line about the in-plane normal and tangent. A displacement finite element model based on Eqs. (4.12)–(4.16) requires the Lagrange interpolation of $(u_0, v_0, \phi_x, \phi_y)$ and Hermite interpolation of w_0. A conforming element will have eight degrees of freedom $(u_0, v_0, w_0, w_{0,x}, w_{0,y}, w_{0,xy}, \phi_x, \phi_y)$ whereas a nonconforming element will have $(u_0, v_0, w_0, w_{0,x}, w_{0,y}, \phi_x, \phi_y)$ seven degrees of freedom per node. In view of the detailed discussion of finite element models of the classical plate theory presented in Section 8, only the salient features of the model are discussed here[52].

9.2. Finite Element Model

The generalized displacements are approximated over an element Ω^e by the expressions

$$\begin{aligned}
u_0(x,y,t) &= \sum_{i=1}^{m} u_i^e(t)\psi_i^e(x,y)\\
v_0(x,y,t) &= \sum_{i=1}^{m} v_i^e(t)\psi_i^e(x,y)\\
w_0(x,y,t) &= \sum_{i=1}^{m} \bar{\Delta}_i^e(t)\varphi_i^e(x,y)\\
\phi_x(x,y,t) &= \sum_{i=1}^{m} X_i^e(t)\psi_i^e(x,y)\\
\phi_y(x,y,t) &= \sum_{i=1}^{m} Y_i^e(t)\psi_i^e(x,y)
\end{aligned} \tag{9.1}$$

where ψ_i^e denote the Lagrange interpolation functions and φ_i^e are the Hermite interpolation functions. Here we chose the same approximation for the in-plane displacements (u_0, v_0) and rotations (ϕ_x, ϕ_y), although one could use different approximations for these two pairs. In the case of the conforming element, the four nodal values associated with w_0 are

$$\bar{\Delta}_1 = w_0, \quad \bar{\Delta}_2 = \frac{\partial w_0}{\partial x}\ , \quad \bar{\Delta}_3 = \frac{\partial w_0}{\partial y}\ , \quad \bar{\Delta}_4 = \frac{\partial^2 w_0}{\partial x \partial y} \tag{9.2}$$

For the nonconforming element, the cross derivative is omitted. The conforming rectangular element with linear interpolation of the in-plane displacements and rotations has eight degrees of freedom per node. The corresponding nonconforming element has seven degrees of freedom per node.

Substitution of Eq. (9.1) into the weak form of Eqs. (3.12)–(3.16) yields the finite element model

$$\begin{aligned}
&\begin{bmatrix}
[K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}]\\
[K^{12}]^T & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}]\\
[K^{13}]^T & [K^{23}]^T & [K^{33}] & [K^{34}] & [K^{35}]\\
[K^{14}]^T & [K^{24}]^T & [K^{34}]^T & [K^{44}] & [K^{45}]\\
[K^{15}]^T & [K^{25}]^T & [K^{35}]^T & [K^{45}]^T & [K^{55}]
\end{bmatrix}
\begin{Bmatrix} \{u^e\}\\ \{v^e\}\\ \{\Delta^e\}\\ \{X^e\}\\ \{Y^e\} \end{Bmatrix}\\
&+ \begin{bmatrix}
[M^{11}] & [0] & [M^{13}] & [M^{14}] & [0]\\
[0] & [M^{22}] & [M^{23}] & [0] & [M^{25}]\\
[M^{13}]^T & [M^{23}]^T & [M^{33}] & [M^{34}] & [M^{35}]\\
[M^{14}]^T & [0] & [M^{34}]^T & [M^{44}] & [0]\\
[0] & [M^{25}]^T & [M^{35}]^T & [0] & [M^{55}]
\end{bmatrix}
\begin{Bmatrix} \{\ddot{u}^e\}\\ \{\ddot{v}^e\}\\ \{\ddot{\Delta}^e\}\\ \{\ddot{X}^e\}\\ \{\ddot{Y}^e\} \end{Bmatrix}\\
&= \begin{Bmatrix} \{F^1\}\\ \{F^2\}\\ \{F^3\}\\ \{F^4\}\\ \{F^5\} \end{Bmatrix}
+ \begin{Bmatrix} \{F^{T1}\}\\ \{F^{T2}\}\\ \{F^{T3}\}\\ \{F^{T4}\}\\ \{F^{T5}\} \end{Bmatrix}
+ \begin{Bmatrix} \{F^{P1}\}\\ \{F^{P2}\}\\ \{F^{P3}\}\\ \{F^{P4}\}\\ \{F^{P5}\} \end{Bmatrix}
\end{aligned} \tag{9.3a}$$

or, in compact form, we can write

$$\sum_{\beta=1}^{5}\sum_{j=1}^{n_\beta}\left(K_{ij}^{\alpha\beta}\Delta_j^\beta + M_{ij}^{\alpha\beta}\ddot{\Delta}_j^\beta + S_{ij}^{\alpha\beta}\Delta_j^\beta\right) - F_i^\alpha = 0, \quad i = 1,2,\ldots,n_\alpha \tag{9.3b}$$

where $\alpha = 1,2,3,4,5$; $n_1 = n_2 = n_4 = n_5 = 4$ and $n_3 = 16$ for the conforming element. The nodal values Δ_j^β, the linear stiffness coefficients $K_{ij}^{\alpha\beta}$, and mass coefficients $M_{ij}^{\alpha\beta}$ are defined by

$$\Delta_j^1 = u_j, \quad \Delta_j^2 = v_j, \quad \Delta_j^3 = \bar{\Delta}_j, \quad \Delta_j^4 = X_j^1, \quad \Delta_j^5 = Y_j^2 \tag{9.4}$$

$$\begin{aligned}
K_{ij}^{1\alpha} &= \int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}N_{1j}^\alpha + \frac{\partial\psi_i}{\partial y}N_{6j}^\alpha\right)dxdy\\
K_{ij}^{2\alpha} &= \int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}N_{6j}^\alpha + \frac{\partial\psi_i}{\partial y}N_{2j}^\alpha\right)dxdy\\
K_{ij}^{3\alpha} &= \int_{\Omega^e}\left[\frac{\partial\varphi_i}{\partial x}\hat{Q}_{1j}^\alpha + \frac{\partial\varphi_i}{\partial y}\hat{Q}_{2j}^\alpha - c_1\left(\frac{\partial^2\varphi_i}{\partial x^2}P_{1j}^\alpha + 2\frac{\partial^2\varphi_i}{\partial x\partial y}P_{6j}^\alpha + \frac{\partial^2\varphi_i}{\partial y^2}P_{2j}^\alpha\right)\right]dxdy\\
K_{ij}^{4\alpha} &= \int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}\hat{M}_{1j}^\alpha + \frac{\partial\psi_i}{\partial y}\hat{M}_{6j}^\alpha + \psi_i\hat{Q}_{1j}^\alpha\right)dxdy\\
K_{ij}^{5\alpha} &= \int_{\Omega^e}\left(\frac{\partial\psi_i}{\partial x}\hat{M}_{6j}^\alpha + \frac{\partial\psi_i}{\partial y}\hat{M}_{2j}^\alpha + \psi_i\hat{Q}_{2j}^\alpha\right)dxdy
\end{aligned} \tag{9.5}$$

$$\begin{aligned}
&M_{ij}^{11} = I_0S_{ij}^0, \quad M_{ij}^{12} = 0, \quad M_{ij}^{13} = -c_1I_3S_{ij}^{0x}, \quad M_{ij}^{14} = J_1S_{ij}^0, \quad M_{ij}^{15} = 0\\
&M_{ij}^{22} = I_0S_{ij}^0, \quad M_{ij}^{23} = -c_1I_3S_{ij}^{0y}, \quad M_{ij}^{24} = 0, \quad M_{ij}^{25} = J_1S_{ij}^0\\
&M_{ij}^{33} = I_0S_{ij}^1 + c_1^2I_6\left(S_{ij}^{xx} + S_{ij}^{yy}\right), \quad M_{ij}^{34} = J_4S_{ji}^{0x}, \ M_{ij}^{35} = J_4S_{ji}^{0y}\\
&M_{ij}^{44} = K_2S_{ij}^0, \quad M_{ij}^{45} = 0, \quad M_{ij}^{55} = K_2S_{ij}^0
\end{aligned} \tag{9.6}$$

$$\begin{aligned}
F_i^1 &= \oint_{\Gamma^e}\left(N_{xx}n_x + N_{xy}n_y\right)\psi_i\ ds, \quad F_i^2 = \oint_{\Gamma^e}\left(N_{xy}n_x + N_{yy}n_y\right)\psi_i\ ds\\
F_i^4 &= \oint_{\Gamma^e}\left(M_{xx}n_x + M_{xy}n_y\right)\psi_i\ ds, \quad F_i^5 = \oint_{\Gamma^e}\left(M_{xy}n_x + M_{yy}n_y\right)\psi_i\ ds\\
F_i^3 &= \int_{\Omega^e}q\varphi_i\ dxdy + \oint_{\Gamma^e}\left(\bar{V}_n\varphi_i + P_{nn}\frac{\partial\varphi_i}{\partial n}\right)\ ds\\
F_i^{T1} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xx}^T + \frac{\partial\psi_i^e}{\partial y}N_{xy}^T\right)\ dxdy, \quad F_i^{T2} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xy}^T + \frac{\partial\psi_i^e}{\partial y}N_{yy}^T\right)\ dxdy\\
F_i^{T4} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}M_{xx}^T + \frac{\partial\psi_i^e}{\partial y}M_{xy}^T\right)\ dxdy, \quad F_i^{T5} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}M_{xy}^T + \frac{\partial\psi_i^e}{\partial y}M_{yy}^T\right)\ dxdy\\
F_i^{P1} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xx}^P + \frac{\partial\psi_i^e}{\partial y}N_{xy}^P\right)\ dxdy, \quad F_i^{P2} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}N_{xy}^P + \frac{\partial\psi_i^e}{\partial y}N_{yy}^P\right)\ dxdy\\
F_i^{P4} &= \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}M_{xx}^P + \frac{\partial\psi_i^e}{\partial y}M_{xy}^P\right)\ dxdy, \quad F_i^{P5} = \int_{\Omega^e}\left(\frac{\partial\psi_i^e}{\partial x}M_{xy}^P + \frac{\partial\psi_i^e}{\partial y}M_{yy}^P\right)\ dxdy
\end{aligned} \tag{9.7}$$

where $\bar{V}_n$ is defined in Eq. (3.21), and P_{nn} is related to P_{xx}, P_{xy}, and P_{yy} in the same way M_{nn} related to M_{xx}, M_{xy}, and M_{yy} [see Eq. (2.18*b*)]. The following notation is used in defining the above coefficients:

$$
\begin{aligned}
S_{ij}^{0} &= \int_{\Omega^e} \psi_i \psi_j \; dxdy, \quad S_{ij}^{0x} = \int_{\Omega^e} \psi_i \frac{\partial \varphi_j}{\partial x} \; dxdy \\
S_{ij}^{0y} &= \int_{\Omega^e} \psi_i \frac{\partial \varphi_i}{\partial y} \; dxdy, \quad S_{ij}^{1} = \int_{\Omega^e} \varphi_i \varphi_j dxdy \\
S_{ij}^{xx} &= \int_{\Omega^e} \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} \; dxdy, \quad S_{ij}^{yy} = \int_{\Omega^e} \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial y} \; dxdy \\
N_{1j}^{1} &= A_{11}\frac{\partial \psi_j}{\partial x} + A_{16}\frac{\partial \psi_j}{\partial y}\;, \quad N_{1j}^{2} = A_{16}\frac{\partial \psi_j}{\partial x} + A_{12}\frac{\partial \psi_j}{\partial y} \\
N_{1j}^{3} &= -c_1\left(E_{11}\frac{\partial^2 \varphi_j}{\partial x^2} + 2E_{16}\frac{\partial^2 \varphi_j}{\partial x \partial y} + E_{12}\frac{\partial^2 \varphi_j}{\partial^2}\right) \\
N_{1j}^{4} &= \hat{B}_{11}\frac{\partial \psi_j}{\partial x} + \hat{B}_{16}\frac{\partial \psi_j}{\partial y}\;, \quad N_{1j}^{5} = \hat{B}_{12}\frac{\partial \psi_j}{\partial x} + \hat{B}_{16}\frac{\partial \psi_j}{\partial y} \\
N_{6j}^{1} &= A_{16}\frac{\partial \psi_j}{\partial x} + A_{66}\frac{\partial \psi_j}{\partial y}\;, \quad N_{6j}^{2} = A_{66}\frac{\partial \psi_j}{\partial x} + A_{26}\frac{\partial \psi_j}{\partial y} \\
N_{6j}^{3} &= -c_1\left(E_{16}\frac{\partial^2 \varphi_j}{\partial x^2} + 2E_{66}\frac{\partial^2 \varphi_j}{\partial x \partial y} + E_{26}\frac{\partial^2 \varphi_j}{\partial^2}\right) \\
N_{6j}^{4} &= \hat{B}_{16}\frac{\partial \psi_j}{\partial x} + \hat{B}_{66}\frac{\partial \psi_j}{\partial y}\;, \quad N_{6j}^{5} = \hat{B}_{26}\frac{\partial \psi_j}{\partial x} + \hat{B}_{66}\frac{\partial \psi_j}{\partial y} \\
N_{2j}^{1} &= A_{12}\frac{\partial \psi_j}{\partial x} + A_{26}\frac{\partial \psi_j}{\partial y}\;, \quad N_{2j}^{2} = A_{26}\frac{\partial \psi_j}{\partial x} + A_{22}\frac{\partial \psi_j}{\partial y} \\
N_{2j}^{3} &= -c_1\left(E_{12}\frac{\partial^2 \varphi_j}{\partial x^2} + 2E_{26}\frac{\partial^2 \varphi_j}{\partial x \partial y} + E_{22}\frac{\partial^2 \varphi_j}{\partial^2}\right) \\
N_{2j}^{4} &= \hat{B}_{12}\frac{\partial \psi_j}{\partial x} + \hat{B}_{26}\frac{\partial \psi_j}{\partial y}\;, \quad N_{2j}^{5} = \hat{B}_{22}\frac{\partial \psi_j}{\partial x} + \hat{B}_{26}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{1j}^{1} &= \hat{B}_{11}\frac{\partial \psi_j}{\partial x} + \hat{B}_{16}\frac{\partial \psi_j}{\partial y}\;, \quad \hat{M}_{2j}^{2} = \hat{B}_{16}\frac{\partial \psi_j}{\partial x} + \hat{B}_{12}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{1j}^{3} &= -c_1\left(\hat{F}_{11}\frac{\partial^2 \varphi_j}{\partial x^2} + 2\hat{F}_{16}\frac{\partial^2 \varphi_j}{\partial x \partial y} + \hat{F}_{12}\frac{\partial^2 \varphi_j}{\partial^2}\right) \\
\hat{M}_{1j}^{4} &= \bar{D}_{11}\frac{\partial \psi_j}{\partial x} + \bar{D}_{16}\frac{\partial \psi_j}{\partial y}\;, \quad \hat{M}_{1j}^{5} = \hat{\bar{D}}_{12}\frac{\partial \psi_j}{\partial x} + \bar{D}_{16}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{6j}^{1} &= \hat{B}_{16}\frac{\partial \psi_j}{\partial x} + \hat{B}_{66}\frac{\partial \psi_j}{\partial y}\;, \quad \hat{M}_{6j}^{2} = \hat{B}_{66}\frac{\partial \psi_j}{\partial x} + \hat{B}_{26}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{6j}^{3} &= -c_1\left(\hat{F}_{16}\frac{\partial^2 \varphi_j}{\partial x^2} + 2\hat{F}_{66}\frac{\partial^2 \varphi_j}{\partial x \partial y} + \hat{F}_{26}\frac{\partial^2 \varphi_j}{\partial^2}\right) \\
\hat{M}_{6j}^{4} &= \bar{D}_{16}\frac{\partial \psi_j}{\partial x} + \bar{D}_{66}\frac{\partial \psi_j}{\partial y}\;, \quad \hat{M}_{6j}^{5} = \bar{D}_{26}\frac{\partial \psi_j}{\partial x} + \bar{D}_{66}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{2j}^{1} &= \hat{B}_{12}\frac{\partial \psi_j}{\partial x} + \hat{B}_{26}\frac{\partial \psi_j}{\partial y}\;, \quad \hat{M}_{2j}^{2} = \hat{B}_{26}\frac{\partial \psi_j}{\partial x} + \hat{B}_{22}\frac{\partial \psi_j}{\partial y} \\
\hat{M}_{2j}^{3} &= -c_1\left(\hat{F}_{12}\frac{\partial^2 \varphi_j}{\partial x^2} + 2F_{26}\frac{\partial^2 \varphi_j}{\partial x \partial y} + F_{22}\frac{\partial^2 \varphi_j}{\partial^2}\right)
\end{aligned}
\tag{9.8}
$$

$$\hat{M}^4_{2j} = \bar{D}_{12}\frac{\partial\psi_j}{\partial x} + \bar{D}_{26}\frac{\partial\psi_j}{\partial y}\ ,\quad \hat{M}^5_{2j} = \bar{D}_{22}\frac{\partial\psi_j}{\partial x} + \bar{D}_{26}\frac{\partial\psi_j}{\partial y} \tag{9.9}$$

$$\begin{aligned}
P^1_{1j} &= E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}\ ,\quad P^2_{1j} = E_{12}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}\\
P^3_{1j} &= -c_1\left(H_{11}\frac{\partial^2\varphi_j}{\partial x^2} + 2H_{16}\frac{\partial^2\varphi_j}{\partial x\partial y} + H_{12}\frac{\partial^2\varphi_j}{\partial^2}\right)\\
P^4_{1j} &= \hat{F}_{11}\frac{\partial\psi_j}{\partial x} + \hat{F}_{16}\frac{\partial\psi_j}{\partial y}\ ,\quad P^5_{1j} = \hat{F}_{12}\frac{\partial\psi_j}{\partial x} + \hat{F}_{16}\frac{\partial\psi_j}{\partial y}\\
P^1_{2j} &= E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}\ ,\quad P^2_{2j} = E_{22}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}\\
P^3_{2j} &= -c_1\left(H_{21}\frac{\partial^2\varphi_j}{\partial x^2} + 2H_{26}\frac{\partial^2\varphi_j}{\partial x\partial y} + H_{22}\frac{\partial^2\varphi_j}{\partial^2}\right)\\
P^4_{2j} &= \hat{F}_{12}\frac{\partial\psi_j}{\partial x} + \hat{F}_{26}\frac{\partial\psi_j}{\partial y}\ ,\quad P^5_{2j} = \hat{F}_{22}\frac{\partial\psi_j}{\partial x} + \hat{F}_{26}\frac{\partial\psi_j}{\partial y}\\
P^1_{6j} &= E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}\ ,\quad P^2_{6j} = E_{26}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}\\
P^3_{6j} &= -c_1\left(H_{16}\frac{\partial^2\varphi_j}{\partial x^2} + 2H_{66}\frac{\partial^2\varphi_j}{\partial x\partial y} + H_{26}\frac{\partial^2\varphi_j}{\partial^2}\right)\\
P^4_{6j} &= \hat{F}_{16}\frac{\partial\psi_j}{\partial x} + \hat{F}_{66}\frac{\partial\psi_j}{\partial y}\ ,\quad P^5_{6j} = \hat{F}_{26}\frac{\partial\psi_j}{\partial x} + \hat{F}_{66}\frac{\partial\psi_j}{\partial y}\\
\hat{Q}^3_{1j} &= \bar{A}_{45}\frac{\partial\varphi_j}{\partial x} + \bar{A}_{55}\frac{\partial\varphi_j}{\partial y}\ ,\quad \hat{Q}^3_{2j} = \bar{A}_{44}\frac{\partial\varphi_j}{\partial x} + \bar{A}_{45}\frac{\partial\varphi_j}{\partial y}\\
\hat{Q}^4_{1j} &= \hat{A}_{55}\psi_j,\quad \hat{Q}^5_{1j} = \hat{Q}^4_{2j} = \hat{A}_{45}\psi_j,\quad \hat{Q}^5_{2j} = \hat{A}_{44}\psi_j
\end{aligned} \tag{9.10}$$

The nonlinear stiffness matrices remain the same as given in Eqs. (9.9)–(9.11).

Note that the stiffness matrix evaluation requires the computation of the second derivatives of the interpolation functions used for the transverse deflection. The isoparametric elements are used for $(u_0, v_0, \phi_x, \phi_y)$ and subparametric formulation is used for w_0. The transformation equations required to numerically evaluate the stiffness coefficients can be found in the textbooks by Reddy[52,53]. The discussion presented there on shear locking with the shear deformable displacement finite element model also applies to the present element in evaluating the shear stiffness coefficients, which should be evaluated using reduced integration.

In the case of the first-order shear deformation plate theory, the resulting finite element models require only C^0–continuity of all generalized displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$, and they are the most economical while accounting for the transverse shear deformation. The theory is discussed in the next section. However, the finite element models will not be discussed[51−−53].

10. Analysis of magnetostrictive plates using FSDT

10.1. Introduction

The grains of certain materials consist of numerous small, randomly oriented magnetic domains that can rotate and align under thew influence of an external magnetic field. The magnetic orientation brings about internal strains in the material. This is known as the *magnetostriction.* This magneto-elastic response is non-linear. A commercially available magnetostrictive material Terfenol-D is an alloy of terbium, iron, and dysprosium.

Laminated composite plates containing magnetostrictive layers are modelled as distributed parameter systems and the magnetostrictive layers are used to control the vibration suppression[56–60]. Velocity feedback with constant gain distributed controller is chosen to achieve vibration suppression.

10.2 Governing equations

Consider a symmetrically laminated composite plate of n layers with the mth and $(n-m+1)$th layers being made of magnetostrictive material, and the remaining $n-2$ layers made of a fiber-reinforced material and having varying fiber orientation θ and symmetrically disposed about the center plane of the plate (see Figure 10.1). The displacement field of the first-order shear deformation theory (FSDT) is of the form

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) \\
v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned} \tag{10.1}
$$

where $(u_0, v_0, w_0, \phi_x, \phi_y)$ are unknown functions to be determined, (u_0, v_0, w_0) denote the displacements of a point on the plane $z = 0$, and which indicate that ϕ_x and ϕ_y are the rotations of a transverse normal about the $y-$ and $x-$axes, respectively.

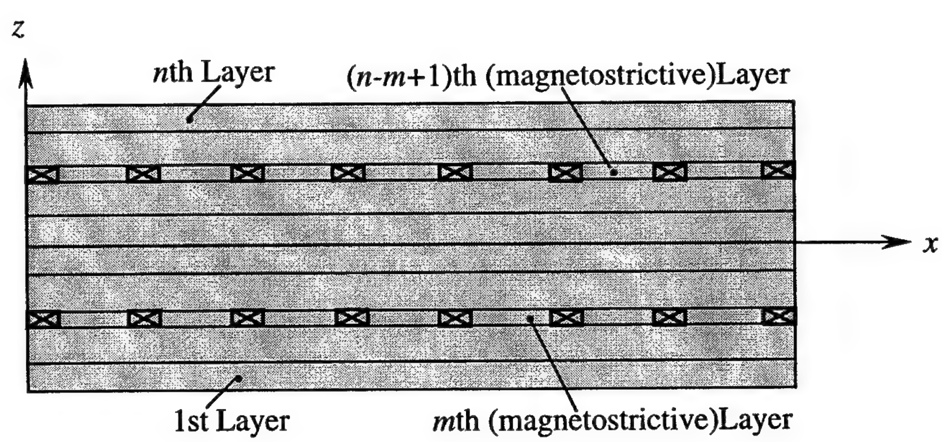

Figure 10.1: Symmetrically laminated plate with magnetostrictive layers.

The linear strains associated with the displacement field (10.1) are

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{yz}^{(0)}\\ \gamma_{xz}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ 0\\ 0\\ \gamma_{xy}^{(1)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial u_0}{\partial x}\\ \frac{\partial v_0}{\partial y}\\ \frac{\partial w_0}{\partial y} + \phi_y\\ \frac{\partial w_0}{\partial x} + \phi_x\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\end{array}\right\} + z\left\{\begin{array}{c}\frac{\partial \phi_x}{\partial x}\\ \frac{\partial \phi_y}{\partial y}\\ 0\\ 0\\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\end{array}\right\} \tag{10.2}$$

The constitutive relations of the kth lamina are

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}^{(k)} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} - \left\{\begin{array}{c}\bar{e}_{31}\\ \bar{e}_{32}\\ \bar{e}_{36}\end{array}\right\}^{(k)} H_z \tag{10.3a}$$

$$\left\{\begin{array}{c}\sigma_{yz}\\ \sigma_{xz}\end{array}\right\}^{(k)} = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}^{(k)} \left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\} \tag{10.3b}$$

where $Q_{ij}^{(k)}$ are the plane stress-reduced stiffnesses and $e_{ij}^{(k)}$ are the magneto-mechanical coupling moduli of the kth lamina [see Eqs. (3.6)–(3.8) and (4.8)].

The governing equations of the first-order theory can be derived using the dynamic version of the principle of virtual displacements. The equations of motion are

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} \tag{10.4a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_y}{\partial t^2} \tag{10.4b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0\frac{\partial^2 w_0}{\partial t^2} \tag{10.4c}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2} \tag{10.4d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2\frac{\partial^2 \phi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2} \tag{10.4e}$$

The natural boundary conditions are of the form

$$N_{nn} - \hat{N}_{nn} = 0\ , \quad N_{ns} - \hat{N}_{ns} = 0\ , \quad Q_n - \hat{Q}_n = 0$$

$$M_{nn} - \hat{M}_{nn} = 0\ , \quad M_{ns} - \hat{M}_{ns} = 0 \tag{10.5}$$

where

$$Q_n \equiv Q_x n_x + Q_y n_y \tag{10.6}$$

Thus the primary and secondary variables of the theory are

$$\begin{array}{rl}\text{primary variables:} & u_n,\ u_s,\ w_0,\ \phi_n,\ \phi_s\\ \text{secondary variables:} & N_{nn},\ N_{ns},\ Q_n,\ M_{nn},\ M_{ns}\end{array} \tag{10.7}$$

The initial conditions of the theory involve specifying the values of the displacements and velocities at time $t = 0$:

$$\begin{aligned} u_n &= u_n^0, \; u_s = u_s^0, \; w_0 = w_0^0, \; \phi_n = \phi_n^0, \; \phi_s = \phi_s^0 \\ \dot{u}_n &= \dot{u}_n^0, \; \dot{u}_s = \dot{u}_s^0, \; \dot{w}_0 = \dot{w}_0^0, \; \dot{\phi}_n = \dot{\phi}_n^0, \; \dot{\phi}_s = \dot{\phi}_s^0 \end{aligned} \tag{10.8}$$

for all points in Ω_0.

10.3 Velocity feedback control

Considering velocity proportional closed-loop feedback control, the magnetic field intensity H_z is expressed in terms of coil current $I(x, y, t)$ as

$$H(x, y, t) = k_c I(x, t) \tag{10.9}$$

and $I(x, y, t)$ is related the velocity $\dot{w}_0$ by

$$I(x, y, t) = c(t)\frac{\partial w_0}{\partial t} \tag{10.10}$$

where k_c is the coil constant, which can be expressed in terms of the coil width b_c, coil radius r_c, and number of turns n_c in the coil by

$$k_c = \frac{n_c}{\sqrt{b_c^2 + 4r_c^2}} \tag{10.11}$$

and $c(t)$ is the control gain.

10.4 Laminate constitutive equations

The force and moment resultants are related to the strains by

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\varepsilon^1\} \end{Bmatrix} - \begin{Bmatrix} \{N\}^M \\ \{M\}^M \end{Bmatrix} \tag{10.12a}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{10.12b}$$

where K is the shear correction factor and

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k) \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2) \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3) \\ (A_{44}, A_{45}, A_{55}) &= \sum_{k=1}^{N} (\bar{Q}_{44}^{(k)}, \bar{Q}_{45}^{(k)}, \bar{Q}_{55}^{(k)})(z_{k+1} - z_k) \end{aligned} \tag{10.13}$$

and the magnetostrictive stress resultants $\{N^M\}$ and $\{M^M\}$ are defined by

$$\begin{aligned}\left\{\begin{array}{c} N_{xx}^M \\ N_{yy}^M \end{array}\right\} &= \sum_{k=m,n-m+1} \int_{z_k}^{z_{k+1}} \left\{\begin{array}{c} \bar{e}_{31} \\ \bar{e}_{32} \end{array}\right\} H_z \, dz \\ &= ck_c \sum_{k=m,n-m+1} \int_{z_k}^{z_{k+1}} \left\{\begin{array}{c} \bar{e}_{31} \\ \bar{e}_{32} \end{array}\right\} \frac{\partial w_0}{\partial t} \, dz \end{aligned} \tag{10.14}$$

$$\begin{aligned}\left\{\begin{array}{c} M_{xx}^M \\ M_{yy}^M \end{array}\right\} &= \sum_{k=m,n-m+1} \int_{z_k}^{z_{k+1}} \left\{\begin{array}{c} \bar{e}_{31} \\ \bar{e}_{32} \end{array}\right\} z H_z \, dz \\ &= ck_c \sum_{k=m,n-m+1} \int_{z_k}^{z_{k+1}} \left\{\begin{array}{c} \bar{e}_{31} \\ \bar{e}_{32} \end{array}\right\} \frac{\partial w_0}{\partial t} z \, dz \\ &\equiv \left\{\begin{array}{c} \mathcal{E}_{31} \\ \mathcal{E}_{32} \end{array}\right\} \frac{\partial w_0}{\partial t} \end{aligned} \tag{10.15}$$

10.5 Analytical solution

Here we seek the Navier solutions for the case of simply supported, symmetric laminates. Towards this end, we write the governing equations in terms of the generalized displacements (w_0, ϕ_x, ϕ_y):

$$KA_{55}\frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) + KA_{44}\frac{\partial}{\partial y}\left(\frac{\partial w_0}{\partial y} + \phi_y\right) + q = I_0\frac{\partial^2 w_0}{\partial t^2} \tag{10.16}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left(D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}\right) + D_{66}\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ &\quad - KA_{55}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - \mathcal{E}_{31}\frac{\partial \dot{w}_0}{\partial x} = I_2\frac{\partial^2 \phi_x}{\partial t^2}\end{aligned} \tag{10.17}$$

$$\begin{aligned}&\frac{\partial}{\partial y}\left(D_{12}\frac{\partial \phi_x}{\partial y} + D_{22}\frac{\partial \phi_y}{\partial y}\right) + D_{66}\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ &\quad - KA_{44}\left(\frac{\partial w_0}{\partial y} + \phi_y\right) - \mathcal{E}_{32}\frac{\partial \dot{w}_0}{\partial y} = I_2\frac{\partial^2 \phi_y}{\partial t^2}\end{aligned} \tag{10.18}$$

The simply supported boundary conditions for the first-order shear deformation plate theory (FSDT) are

$$\begin{aligned}&\phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(0,y,t) = 0, \quad , \quad \phi_y(a,y,t) = 0 \\ &w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \\ &M_{xx}(0,y,t) = 0, \quad M_{xx}(a,y,t) = 0, \quad M_{yy}(x,0,t) = 0, \quad M_{yy}(x,b,t) = 0\end{aligned} \tag{10.19}$$

The boundary conditions in (10.19) are satisfied by the following expansions

$$w_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}(t)\sin\alpha x \sin\beta y \tag{10.20a}$$

$$\phi_x(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_{mn}(t)\cos\alpha x \sin\beta y \tag{10.20b}$$

$$\phi_y(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Y_{mn}(t)\sin\alpha x \cos\beta y \tag{10.20c}$$

The mechanical load and magnetostrictive moments are also expanded in double Fourier sine series

$$q(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{mn}(t)\sin\alpha x \sin\beta y \tag{10.21a}$$

$$M_{xx}^{M}(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M_{mn}^{1}(t)\sin\alpha x \sin\beta y \tag{10.21b}$$

$$M_{yy}^{M}(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M_{mn}^{2}(t)\sin\alpha x \sin\beta y \tag{10.21c}$$

where, for example,

$$Q_{mn}(t) = \frac{4}{ab}\int_0^a\int_0^b q(x,y,t)\ \sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{b}\ dxdy \tag{10.22a}$$

$$M_{mn}^{1}(t) = \frac{4}{ab}\int_0^a\int_0^b M_{xx}^{M}(x,y,t)\ \sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{b}\ dxdy \tag{10.22b}$$

Substitution of Eqs. (10.20)–(10.22) into Eqs. (10.16)–(10.18) yields the equations

$$\begin{aligned}&\begin{bmatrix}\hat{s}_{33} & \hat{s}_{34} & \hat{s}_{35}\\ \hat{s}_{34} & \hat{s}_{44} & \hat{s}_{45}\\ \hat{s}_{35} & \hat{s}_{45} & \hat{s}_{55}\end{bmatrix}\begin{Bmatrix}W_{mn}\\ X_{mn}\\ Y_{mn}\end{Bmatrix} + \begin{bmatrix}0 & 0 & 0\\ \hat{c}_{12} & 0 & 0\\ \hat{c}_{13} & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{W}_{mn}\\ \dot{X}_{mn}\\ \dot{Y}_{mn}\end{Bmatrix}\\ &+\begin{bmatrix}\hat{m}_{33} & 0 & 0\\ 0 & \hat{m}_{44} & 0\\ 0 & 0 & \hat{m}_{55}\end{bmatrix}\begin{Bmatrix}\ddot{W}_{mn}\\ \ddot{X}_{mn}\\ \ddot{Y}_{mn}\end{Bmatrix} = \begin{Bmatrix}Q_{mn}\\ 0\\ 0\end{Bmatrix}\end{aligned} \tag{10.23}$$

where $\hat{s}_{ij}$, c_{ij}, and $\hat{m}_{ij}$ are defined by

$$\begin{aligned}&\hat{s}_{33} = K(A_{55}\alpha^2 + A_{44}\beta^2)\\ &\hat{s}_{34} = KA_{55}\alpha,\quad \hat{s}_{35} = KA_{44}\beta,\quad \hat{s}_{44} = (D_{11}\alpha^2 + D_{66}\beta^2 + KA_{55})\\ &\hat{s}_{45} = (D_{12}+D_{66})\alpha\beta,\quad \hat{s}_{55} = (D_{66}\alpha^2 + D_{22}\beta^2 + KA_{44})\\ &\hat{c}_{12} = \alpha\mathcal{E}_{12},\quad \hat{c}_{13} = \beta\mathcal{E}_{13}\end{aligned} \tag{10.24a}$$

$$\hat{m}_{33} = I_0,\quad \hat{m}_{44} = I_2,\quad \hat{m}_{55} = I_2 \tag{10.24b}$$

where the magnetostrictive coefficients $\mathcal{E}_{12}$, and $\mathcal{E}_{13}$ are defined in Eq. (10.18).

For vibration control, we assume $q = 0$ and solution of the ordinary differential equations in Eq. (10.26) in the form

$$W_{mn}(t) = W_0 e^{\lambda t}, \quad X_{mn}(t) = X_0 e^{\lambda t}, \quad Y_{mn}(t) = Y_0 e^{\lambda t} \tag{10.25}$$

and obtain, for non-trivial solution, the result

$$\begin{vmatrix} \bar{S}_{33} & \bar{S}_{34} & \bar{s}_{35} \\ \bar{S}_{43} & \bar{S}_{44} & \bar{S}_{45} \\ \bar{S}_{53} & \bar{S}_{54} & \bar{S}_{55} \end{vmatrix} = 0 \tag{10.26}$$

where

$$\bar{S}_{ij} = \hat{S}_{ij} + \lambda \hat{C}_{ij} + \lambda^2 \hat{M}_{ij} \tag{10.27}$$

where $i, j = 3, 4, 5$. This equation gives three sets of eigenvalues. The lowest one corresponds to the transverse motion. The eigenvalue can be written as $\lambda = -\alpha + i\omega_d$, so that the damped motion is given by

$$w_0(x, y, t) = \frac{1}{\omega_d} e^{-\alpha t} \sin \omega_d t \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{10.28}$$

In arriving at the last solution, the following initial conditions are used:

$$w_0(x, y, 0) = 0, \quad \dot{w}_0(x, y, 0) = 1, \quad \phi_x(x, y, 0) = 0, \quad \dot{\phi}_x(x, y, 0) = 0$$

$$\phi_y(x, y, 0) = 0, \quad \dot{\phi}_y(x, y, 0) = 0 \tag{10.29}$$

10.6. Numerical results and discussion

Numerical studies were carried out to analyze damped natural frequencies, damping coefficients, and the vibration suppression time, using the three theories. Different lay-ups were used to show the influence of the position of magnetostrictive layer on the vibration suppression time. A time ratio relation between the thickness of the layers and the distance to the neutral axis of the laminated composite beam is found. All values of the material and structural constants are indicated in the tables.

The numerical values of various coefficients (namely, the inertial and magnetostrictive coefficients) based on different lay-ups and material properties [CFRP, Graphite-Epoxy (AS), Glass-Epoxy and Boron-Epoxy] are listed in Tables 10.1 and 10.2. Table 10.2 also shows the damping coefficients and natural frequencies for different materials and lay-ups. The damping and frequency parameters for transverse modes $n = 1$ to $n = 5$ are shown in Table 10.3, and they are compared with the results obtained by Murthy et al. [56] using the Euler-Bernoulli beam theory (EBT). Only in the higher modes there is some difference between the numerical results predicted by the three theories. Table 10.4 shows the influence of the position of the magnetostrictive layer in the z-direction and the influence of the lamination scheme in the damping and frequency parameters. The value of α increases when the magnetostrictive layer is located further away from the x-axis, indicating faster vibration suppression. The lay-up $[m/90_4]_s$ represents the softest beam and the lay-up $[m/0_4]_s$ the stiffest beam.

Table 10.1. Coefficients for different Lay-ups and materials.

Material	Lay-up	D_{11} (10^3)	F_{11} (10^{-2})	H_{11} (10^{-7})	A_{55} (10^9)	D_{55} (10^2)	F_{55} (10^{-3})
CFRP	$[\pm45/m/0/90]_S$	3.739	5.246	9.333	6.620	5.185	6.902
	$[45/m/-45/0/90]_S$	3.552	4.891	8.793	6.620	6.179	8.792
	$[m/\pm45/0/90]_S$	3.303	4.069	6.679	6.620	7.506	13.168
	$[m/90_4]_S$	1.432	2.567	5.063	6.620	7.506	13.168
	$[m/0_4]_S$	7.015	7.927	11.189	6.620	7.506	13.168
Gr.-Ep (AS)	$[\pm45/m/0/90]_S$	3.954	5.629	10.053	7.974	6.399	8.881
Gl.-EP	$[\pm45/m/0/90]_S$	2.535	3.700	6.589	7.614	6.173	8.384
Br.-Ep	$[\pm45/m/0/90]_S$	5.730	8.259	14.865	7.066	5.634	7.569

Data: **CFRP :** E_{11}=138.6 Gpa E_{22}=8.27 Gpa G_{13}=G_{23}=0.6 E_{22} G_{12}=4.12 Gpa ν_{12}=0.26 ρ=1824 kg.m^{-3}
Graphite-Epoxy (AS) : E_{11}=137.9 GPa E_{22}=8.96 Gpa G_{12}=G_{13}=7.10 Gpa G_{23}=6.21 Gpa ν_{12}=0.30 ρ=1450 kg.m^{-3}
Glass-Epoxy : E_{11}=53.78 Gpa E_{22}=17.93 Gpa G_{12}=G_{13}=8.96 Gpa G_{23}=3.45 Gpa ν_{12}=0.25 ρ=1900 kg.m^{-3}
Boron-Epoxy : E_{11}=206.9 Gpa E_{22}=20.69 Gpa G_{12}= G_{13}=6.9 Gpa G_{23}=4.14 Gpa ν_{12}=0.30 ρ=1950 kg.m^{-3}

Table 10.2. Inertial and magnetostrictive coefficients and the parameters α and ω_{dn}.

Material	Lay-up	I_0	I_2 (10^{-4})	I_4 (10^{-9})	I_6 (10^{-14})	$-\beta$	$-\varepsilon$ (10^{-4})	$-\alpha \pm \omega_{dn}$(rad/s)
CFRP	$[\pm45/m/0/90]_S$	33.092	2.461	2.907	4.508	22.128	1.438	3.30±104.85
	$[45/m/-45/0/90]_S$	33.092	3.352	4.600	7.084	30.979	3.872	4.62±102.15
	$[m/\pm45/0/90]_S$	33.092	4.540	8.521	17.171	39.830	8.165	5.94±98.42
	$[m/90_4]_S$	33.092	4.540	8.521	17.171	39.830	8.165	5.94±64.65
	$[m/0_4]_S$	33.092	4.540	8.521	17.171	39.830	8.165	5.94±143.57
Gr.-Ep	$[\pm45/m/0/90]_S$	30.100	2.196	2.471	3.696	22.128	1.438	3.63±113.06
Gl.-EP	$[\pm45/m/0/90]_S$	33.700	2.514	2.995	4.674	22.128	1.438	3.24±85.54
Br.-Ep	$[\pm45/m/0/90]_S$	34.100	2.550	3.054	4.782	22.128	1.438	3.20±127.90

Table 10.3. Damping and frequency parameters due to the transverse modes.

$-\alpha \pm \omega_{dn}$ (rad/s) - Lay-up $[\pm45/m/0/90]_S$				
Mode	**Murty et al**	**EBT**	**TBT**	**RBT**
1	3.29±104.88	3.30±104.85	3.30±104.82	3.30±104.82
2	13.19±419.50	13.20±419.37	13.17±418.90	13.16±418.80
3	29.70±943.88	29.68±943.40	29.53±941.05	29.48±940.52
4	52.86±1678.83	52.73±1676.72	52.27±1669.32	52.10±1667.68
5	82.59±2621.87	82.34±2619.02	81.22±2601.04	80.80±2597.09

Data: **CFRP :** E_{11}=138.6 Gpa E_{22}=8.27 GPa G_{12}=4.12 GPa G_{13}=G_{23}=0.6 E_{22} ν_{12}=0.26 ρ=1824 kg.m^{-3}
Magnetostrictive layer : E_m=26.5 GPa ρ_m=9250 kg.m^{-3} d_k=1.67x10^{-8}m/A c(t).R_c=10^4 ν_m=0 a=1 m

Table 10.4. Damping and frequency parameters for different lay-ups.

$-\alpha \pm \omega_{dn}$ (rad/s) - mode 1				
Lay-up	**Murty et al**	**EBT**	**TBT**	**RBT**
$[45/m/-45/0/90]_S$	4.60±102.17	4.62±102.15	4.62±102.12	4.62±102.11
$[m/\pm45/0/90]_S$	5.90±98.44	5.94±98.42	5.94±98.39	5.93±98.38
$[m/90_4]_S$	5.90±64.65	5.94±64.65	5.94±64.64	5.94±64.64
$[m/0_4]_S$	5.90±143.58	5.94±143.57	5.93±143.49	5.93±143.44

A comparision of the fundamental transverse and axial modes, obtained using the Timoshenko beam theory (TBT) and Reddy third-order beam theory (RBT) show (results are not included here due to space limitations) that the Reddy third-order beam theory give lower values for natural frequency. The comparision of the uncontrolled and controlled motion at the mid point of the beam is shown in Figures 10.2–10.5 for the first mode. These figures show that the vibration suppression time decreases when the distance to the neutral axis is increased, and it remains nearly the same in the laminates with different stiffness.

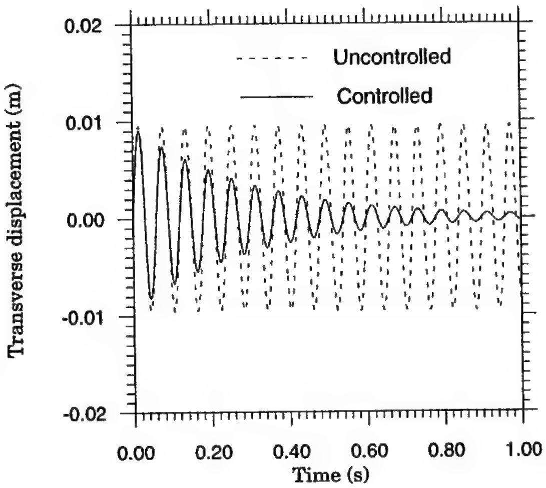

Figure 10.2: Comparison of original and controlled motion at the midpoint of the beam ($[45/-45/m/0/90]_S$).

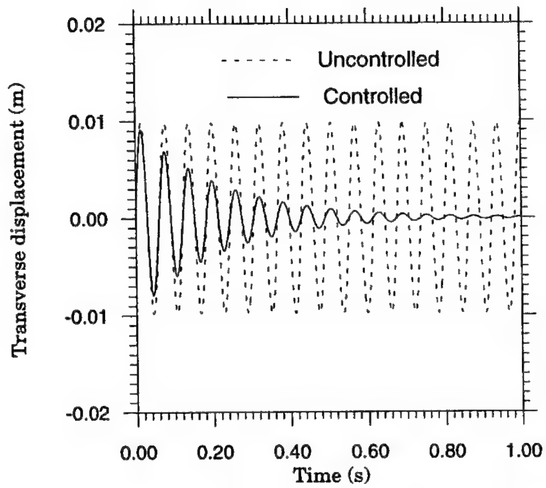

Figure 10.3: Comparison of original and controlled motion at the midpoint of the beam ($[45/m/-45/0/90]_S$).

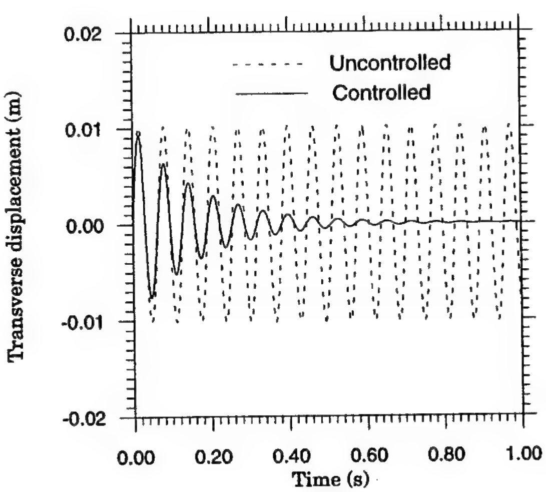

Figure 10.4: Comparison of original and controlled motion at the midpoint of the beam ($[m/45/-45/0/90]_S$).

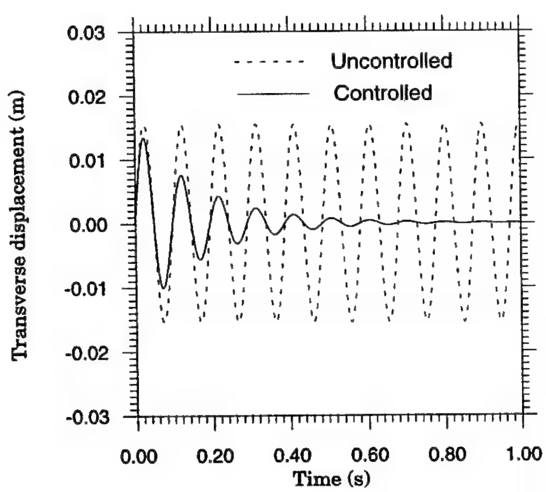

Figure 10.5: Comparison of original and controlled motion at the midpoint of the beam ($[m/90_4]_S$).

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